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COMPLETENESS IN TOPOLOGICAL VECTOR LATTICES*

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One of the early successes of the Lebesgue integral was the observation that the space $L_2(a, b)$ of square-summable functions on the interval (a, b) is complete. The initial historical interest in this fact was the discovery that it constitutes the main point in the proof that if $\{\phi_n\}$ is an orthonormal basis for $L_2(a, b)$, then if $\{a_n\}$ is any sequence of real numbers for which $\sum_{n=1}^{\infty} a_n^2 < \infty$, it follows that $\sum_{n=1}^{\infty} a_n\phi_n$ is the Fourier expansion, with respect to this basis, of a function $f \in L_2(a, b)$, and the sequence $f_n = \sum_{k=1}^{n} a_k\phi_k$ converges to f in the space $L_2(a, b)$; *i.e.*,

$$\lim_{n\to\infty} \int_a^b [f(x) - f_n(x)]^2 dx = 0.$$

For, in order to have this result, it is only necessary to show that $\{f_n\}$ is a Cauchy sequence in $L_2(a, b)$. It then follows from the completeness of $L_2(a, b)$ that the limit of $\{f_n\}$ in $L_2(a, b)$ is the required f.

Although the proof of the completeness of $L_2(a, b)$ is not difficult, its interest is emphasized by the large number of eminent mathematicians (von Neumann, Weyl, *etc.*) who wrote proofs of their own. The proof most often quoted nowadays consists in showing that if $\{g_n\}$ is a Cauchy sequence in $L_2(a, b)$ then it has a subsequence which converges almost everywhere to a function f, thus locating the precious object. It is then shown that $f \in L_2(a, b)$, and finally that f is the limit of $\{g_n\}$ in $L_2(a, b)$.

This argument applies to all L_p spaces, $p \geqq 1$, and to Orlicz spaces, which we need not define here, as well.

Köthe has defined an important class of sequence spaces and has developed their properties in a series of interesting papers, [1], [2], [3]. Dieudonné [4] has extended the definition to spaces of locally summable functions on a locally compact space X (which is, however, restricted to be the union of a countable set of compact subspaces) with a specified nonnegative Radon measure [7] on X. These spaces admit various topologies in a natural way, and are complete for all these topologies. We shall be concerned in this paper only with spaces of real functions.

A very interesting theorem has been given by Nakano [5] who showed that, in a certain kind of topological vector lattice, a sort of completeness in terms of the order relation implies topological completeness.

Now, Köthe spaces à la Dieudonné, even without the countability restriction, fall within the Nakano framework and are easily seen to satisfy his type of order completeness. If this exposition can make any claim to making a new contribution, it is the observation of this connection, which establishes the completeness property for a wider class of Köthe spaces than previously considered.

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Nakano's theorem. We merely present the definitions needed to state the theorem. The proof will not be given here. Let X be a real vector space (we assume the definition is known). Let $P \subset X$ be a subset of X such that $x, y \in P$ and $a \in R$, $a \geqq 0$, (R is the space of real numbers) implies $x+y \in P$ and $ax \in P$. In particular, the zero vector θ of X belongs to P. Let $-P$ be the set of all $x \in X$ for which $-x \in P$; we suppose that θ is the only element belonging to both P and $-P$. Then P is called a positive cone in X. P defines an order relation in X by $x \geqq y$ if and only if $x-y \in P$. A vector space X together with a positive cone $P \subset X$ is called a *vector lattice* if every pair $x, y \in X$ has a least upper bound; it is designated as $x \cup y$. It then easily follows that every $x, y \in X$ has a greatest lower bound which is designated as $x \cap y$. A vector lattice is called *conditionally complete* if every set which has an upper bound has a least upper bound. An important fact about a vector lattice X is that every $x \in X$ has a canonical representation as the difference $x = x^+ - x^-$ of two elements in P; *i.e.*, $x^+, x^- \geqq \theta$. In this representation, $x^+ = x \cup \theta$ and $x^- = -(x \cap \theta)$. For every $x \in X$, we can define the absolute value of x as $|x| = x^+ + x^-$. (General references on vector lattices where these facts and many others are proved are [6], [7]).

A *convex topological vector space* X is a vector space with a topology such that the mappings

α) $(x, y) \to x+y$ *of* $X \times X$ *into* X,
β) $x \to -x$ *of* X *into* X,
γ) $(a, x) \to ax$ *of* $R \times X$ *into* X,

are continuous, and where for every $x, y \in X$, $y \neq x$, and open G, with $x \in G$, there is an open convex H, with $H \subset G$, $x \in H$, and $y \notin H$. In such a space, the topology may also be defined in terms of certain sets, containing the identity θ, which we call balls.

We now give the definition of a convex topological vector space in this form. In a vector space X, a *ball* U is a set which is convex, symmetric, and absorbing; *i.e.*,

a) *if* $x, y \in U$ *and* $0 \leqq a \leqq 1$, *then* $ax+(1-a)y \in U$,
b) *if* $x \in U$ *then* $-x \in U$,
c) *if* $x \in X$, *then there are* $y \in U$ *and* $a \in R$ *for which* $x = ay$.

Now, the topology in a convex topological vector space is determined by a collection $\mathfrak{U}$ of balls for which

i) *if* $x \neq \theta$, *there is a* $U \in \mathfrak{U}$ *with* $x \notin U$,
ii) *if* $U, V \in \mathfrak{U}$, *there is a* $W \in \mathfrak{U}$ *with* $W \subset U \cap V$,
iii) *if* $U \in \mathfrak{U}$, *there is a* $V \in \mathfrak{U}$ *with* $V+V \subset U$.

(The set $V+V$ consists of all $x+y$ where $x, y \in V$).

A set $G \subset X$ is open if for every $x \in G$ there is a $U \in \mathfrak{U}$ with $U_x \subset G$, where U_x is the set of all $y = z+x$, $z \in U$. In other words, G is open means that G is the union of a family of translations of sets in $\mathfrak{U}$. It is a routine task to show

that the topology defined in this way satisfies conditions $\alpha)$, $\beta)$, $\gamma)$. (As a general reference for topological vector spaces we can cite only [8]).

Having defined vector lattice and convex topological vector space, we can now define *convex topological vector lattice*. Let X be a vector lattice with a set $\mathfrak{U}$ of balls satisfying i), ii), iii), and which conform to the order relation in X. Specifically,

iv) *for every* $U\in\mathfrak{U}$, *if* $x\in U$ *and* $|y|\leqq|x|$ *then* $y\in U$.

If we define a set $A\subset X$ to be *order-closed* if $x\in A$ and $|y|\leqq|x|$ implies $y\in A$ the condition iv) simply asserts that every U is order-closed. A convex topological vector lattice is a vector lattice and a topological vector space with topology given by a set $\mathfrak{U}=[U]$ of order-closed balls.

A Nakano space is a special kind of convex topological vector lattice. This involves specialization of the kind of vector lattice allowed as well as of the admissible set $\mathfrak{U}$ of balls. The vector lattice is assumed to be conditionally complete. The restriction on the set of $U\in\mathfrak{U}$ requires some discussion. A *directed set* D is a partly ordered set with order relation "$\geqq$" such that for every $\lambda, \lambda'\in D$ there is $\lambda''\in D$ with $\lambda''\geqq\lambda$, $\lambda''\geqq\lambda'$. A *net* in X is a function on a directed set D with values $x_\lambda\in X$. A net is *increasing* if $\lambda'\geqq\lambda$ implies $x_{\lambda'}\geqq x_\lambda$. A set $U\subset X$ is a *Nakano ball* if it is an order-closed ball, and if for every increasing net x_λ with values in $U\cap P$, which has an upper bound in X, the least upper bound, $x=\sup_\lambda x_\lambda$, is in U. A *Nakano space* is a convex topological vector lattice which is conditionally complete as a vector lattice and for which the sets in $\mathfrak{U}$ are Nakano balls.

The theorem of Nakano connects two kinds of completeness, topological completeness and monotone completeness, notions which we now define.

In a topological vector space X, a set $S\subset X$ is *bounded* if it is absorbed by every $U\in\mathfrak{U}$. That is, if for every $U\in\mathfrak{U}$ there is an $a\in R$ such that $S\subset aU$. (For any $A\subset X$, aA means, as always, the set of all ax, $x\in A$). A Nakano space X is called *monotone complete* if every increasing net in P which is topologically bounded is order bounded. In other words, if x_λ is an increasing net in P whose value set is topologically bounded, then x_λ has an upper bound so that, since X is conditionally complete, $\sup_\lambda x_\lambda$ exists.

A net x_λ in a topological vector space is called a *Cauchy net* if for every $U\in\mathfrak{U}$ there is a ξ such that $\lambda\geqq\xi$, $\lambda'\geqq\xi$ implies $x_\lambda-x_{\lambda'}\in U$. x is the limit of a net x_λ if for every $U\in\mathfrak{U}$ there is ξ such that $\lambda\geqq\xi$ implies $x-x_\lambda\in U$. The space X is *topologically complete* if every Cauchy net in X has a limit.

We may now state

THEOREM 1. (Nakano). *If a Nakano space is monotone complete then it is topologically complete.*

We shall not prove this theorem but feel it is fair to say that the reader who has come this far should be able to read the original proof in [5].

Köthe spaces. Let S be a locally compact Hausdorff space, and let μ be a nonnegative Radon measure on S. Then $\mu(C)<\infty$ for every compact $C\subset S$. Let Ω be the vector lattice of equivalence classes of *locally summable* functions on S. The function f is locally summable if, for every compact $C\subset S$, $\int_C |f|\,d\mu<\infty$, and f is equivalent to g if, for every compact $C\subset S$, $\int_C |f-g|\,d\mu=0$. For convenience, we speak of functions themselves as elements of Ω, when there is no danger of confusion, but mean the equivalence classes to which the functions belong.

Köthe spaces come as pairs K, K^* of subspaces of Ω, which we shall call Köthe duals of each other and which we now define. Let A be any set of functions in Ω. We associate with A a Köthe space K as follows:

$f\in K$ if and only if $f\in\Omega$ and $\int_S |fg|\,d\mu<\infty$ for every $g\in A$. If f, $g\in K$, an immediate calculation shows that $f+g\in K$ and $af\in K$ for every $a\in R$. Hence, K is a vector space which is a subspace of Ω. Now, if $f\geqq\theta$ means $f(x)\geqq 0$ on a set whose complement meets every compact subset of S in a set of measure 0, then Ω is a vector lattice. It is clear from the definition that if $f\in K$ and $|f|\geqq|g|$ then $g\in K$. This, together with the fact that K is a vector space implies, since $|f|+|g|\geqq|f|\cup|g|$, that K is a vector lattice. Corresponding to K is the set K^* of all $f\in\Omega$ such that $\int_S |fg|\,d\mu<\infty$ for all $g\in K$. Then K^* is also a vector lattice. Since $\int_S |fg|\,d\mu<\infty$ for all $f\in A$ and $g\in K$ it follows that $A\subset K^*$. The vector lattice K^* is the Köthe dual of K. We show that K is the Köthe dual of K^*; *i.e.*, $(K^*)^*=K$. First, if $f\in K$ then, for every $g\in K^*$, $\int_S |fg|\,d\mu<\infty$, so that $f\in(K^*)^*$. Next, if $f\in(K^*)^*$, then $\int_S |fg|\,d\mu<\infty$ for every $g\in K^*$. In particular, $\int_S |fg|\,d\mu<\infty$ for every $g\in A$, so that $f\in K$. This proves $(K^*)^*=K$.

The Köthe pair K, K^* is thus a pair of vector lattices, contained in Ω, each of which is the Köthe dual of the other. As an example, let $S=R$, the space of real numbers, and let Ω be the vector lattice of locally summable real functions on R. Then the vector lattice of summable functions on R and the vector lattice of bounded measurable functions on R are a Köthe pair K, K^*. (Thus, the space of summable functions is "reflexive" as a Köthe space, since every Köthe space has this property; while, as a Banach space with norm $\|f\|=\int_R |f|\,dt$, it is not reflexive). If K is the set of pth power integrable functions on R, then K^* is the set of qth power integrable functions, where $p^{-1}+q^{-1}=1$. In addition, various other spaces of functional analysis are included in the Köthe system.

Thus far, K is merely a vector lattice. We now show how K may be given many topologies, for each of which it becomes a convex topological vector lattice. These topologies are defined via the Köthe duality between K and K^*. A set $A\subset K^*$ is called *admissible* if for every $f\in K$,

$$\sup_{g\in A}\int_S |fg|\,d\mu<\infty.$$

In particular, the definition of K^* assures that every set in K^* consisting of a single element is admissible. Now, every admissible $A\subset K^*$ and every $k>0$

determine a ball in K. This is the set U_k of all $f\in K$ for which $\sup_{g\in A}\int_S|fg|\,d\mu \leqq k$. That U_k is a ball is easily established. For, if $\sup_{g\in A}\int_S|fg|\,d\mu\leqq k$, $\sup_{g\in A}\int_S|f'g|\,d\mu\leqq k$, and $0\leqq a\leqq 1$, then $\sup_{g\in A}\int_S|hg|\,d\mu\leqq k$, where $h=af+(1-a)f'$, so that U_k is convex. If $\sup_{g\in A}\int_S|fg|\,d\mu\leqq k$ then $\sup_{g\in A}\int_S|-fg|\,d\mu \leqq k$, so that U_k is symmetric. For every $f\in K$, $\sup_{g\in A}\int_S|fg|\,d\mu=\alpha<\infty$. Now, $\alpha>0$, if there is a $g\neq\theta$ in A and if $f\neq\theta$. Hence, $k\alpha^{-1}f\in U_k$. If A consists only of θ, then $U_k=K$. If $f=\theta$, then $f\in U_k$ so that U_k is absorbing, in any case.

We next show that a collection $\mathfrak{A}$ of admissible sets in K^* defines a convex topology in K if:

I) *every $g\in K^*$ belongs to at least one $A\in\mathfrak{A}$,*
II) *if A, $A'\in\mathfrak{A}$, then $A\cup A'\in\mathfrak{A}$.*

Proof. (i) Let $f\in K$, $f\neq\theta$. There is a $g\in K^*$ with $\int_S|fg|\,d\mu>1$. But there is an $A\in\mathfrak{A}$ with $g\in A$. Thus, $\sup_{g\in A}\int_S|fg|\,d\mu>1$ so that $f\notin U_1(A)$.

(ii) Let r, $s>0$, and let $t=\min(r, s)$ and A, $A'\in\mathfrak{A}$. Then $U_t(A\cup A')\subset U_r(A)\cap U_s(A')$. For, if

$$\sup_{g\in A\cup A'}\int_S|fg|\,d\mu\leqq t,$$

then $\sup_{g\in A}\int_S|fg|\,d\mu\leqq r$ and $\sup_{g\in A'}\int_S|fg|\,d\mu\leqq s$.

(iii) Let $A\in\mathfrak{A}$ and $k>0$. Then $U_{k/2}(A)+U_{k/2}(A)\subset U_k(A)$. We leave this last simple verification to the reader.

A topology in K given in this way by a collection $\mathfrak{A}$ of admissible sets satisfying (I) and (II) is called a *Köthe topology* and the space K with any such topology is called a *topological Köthe space.* Furthermore, the balls U_k are order-closed in the vector lattice K. For, if $\sup_{g\in A}\int_S|fg|\,d\mu\leqq k$ and $|f'|\leqq|f|$ then $\sup_{g\in A}\int_S|f'g|\,d\mu\leqq k$. Hence, every topological Köthe space is a convex topological vector lattice.

Examples of collections $\mathfrak{A}$ of admissible sets satisfying (I) and (II) are the collection of all finite subsets of K^* and the collection of all admissible subsets of K^*. These are the extreme cases, giving the coarsest and finest Köthe topologies for K, respectively.

We now prove:

THEOREM 2. *Every topological Köthe space is a Nakano space.*

Proof. We first show that K is conditionally complete. We need only consider positive elements for this. Let $B\subset K$ be a bounded set of positive elements, and let $u\in K$ be an upper bound. Since Ω is conditionally complete, B has a least upper bound $f\in\Omega$, $f\leqq u$. But K is order-closed, since $\int_S|hg|\,d\mu<\infty$ for every $g\in K^*$ and $|h'|\leqq|h|$ implies $\int_S|h'g|\,d\mu<\infty$ for every $g\in K^*$. Hence $f\in K$, and so K is conditionally complete.

It remains only for us to show that every U_k is a Nakano ball. For this, let f_λ be an increasing net of positive elements in U_k, with an upper bound. Let f

be the least upper bound of f_λ. Then, for every $g \in K^*$, $\int_S |fg|\,d\mu = \sup_\lambda \int_S |f_\lambda g|\,d\mu$ so that $\sup_{g\in A} \int_S |f_\lambda g|\,d\mu \leqq k$, for every λ, implies $\sup_{g\in A} \int_S |fg|\,d\mu \leqq k$. This completes the proof.

Completeness of Köthe spaces. We are now ready to show that every topological Köthe space is topologically complete. Since every topological Köthe space is a Nakano space, it suffices, by Nakano's theorem, to show that K is monotone complete. We proceed to do this. (No countability assumptions are made for the locally compact space S on which the functions in Ω are defined.) Let $\mathfrak{A} = [A]$ be a collection of admissible sets in K^* satisfying (I) and (II). Let f_λ be an increasing net of positive elements in K which is bounded in the associated topology. This means that for every $A \in \mathfrak{A}$ there is a constant $M(A)$ such that $\int_S |f_\lambda g|\,d\mu < M(A)$ for every λ and every $g \in A$. In particular, for every $g \in K^*$, there is a constant $M(g)$ such that $\sup_\lambda \int_S |f_\lambda g|\,d\mu < M(g)$. Let $C \subset S$ be compact, and χ_C the characteristic function of C. We then have $\sup_\lambda \int_S |f_\lambda \chi_C|\,d\mu < M(\chi_C)$, or $\sup_\lambda \int_C f_\lambda\,d\mu < M(\chi_C)$. The restrictions of the functions f_λ to C, call them $f_\lambda | C$, then converge almost everywhere to a function $f | C$ whose domain is C. If C_1 and C_2 are compact, then $f | C_1$ and $f | C_2$ agree almost everywhere on $C_1 \cap C_2$. The collection $\{f | C\}$ of functions thus defines a function f on S (of course, modulo a set which meets every compact set in a set of measure 0). The function f belongs to Ω and is the least upper bound of f.

It remains only to show that $f \in K$. For every compact $C \subset S$ and $g \in K^*$, $\int_C |fg|\,d\mu = \sup_\lambda \int_C |f_\lambda g|\,d\mu \leqq \sup_\lambda \int_S |f_\lambda g|\,d\mu \leqq M(g)$. But, $\int_S |fg|\,d\mu = \sup_C \int_C |fg|\,d\mu \leqq M(g)$ so that $f \in K$. This completes the proof that K is monotone complete, and we have:

THEOREM 3. *Every topological Köthe space is topologically complete.*

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POSTULATES FOR COMMUTATIVE GROUPS*

MARLOW SHOLANDER, Carnegie Institute of Technology

In a set of postulates one does not expect to find the laws of classical logic, the rules of grammar, or definitions of various words. Often omitted are the statements that equality is an equivalence relation and that, in a binary operation, one is free to replace elements by equal elements. For the purposes of this paper it seems natural to go further and ignore, for operations introduced elsewhere, the postulated closure of the system under these operations. We count as postulates only those which are identities or those which have one conditional equation as a consequence of another.

This paper gives two one-postulate systems which characterize commutative groups. The author's interest in one-postulate systems began with alternation groupoids ([2], Postulate II), a generalization of Abelian quasigroup which independently caught the interest of O. Frink [1]. Since then, parthenogenetic postulates for Boolean groups and Boolean algebras have been found [3]. Single postulates for ordinary groups have not yet been found.

1. Ordinary operations. Consider a set S closed under an operation denoted by multiplication. To each element x in S corresponds an element x' in S. The following postulate holds:

(G) $$\text{If } (aa')b' = (rs')t', \text{ then } b = (tr')s.$$

From the case $r=s=a$ and $t=b$ we have

(1.1) $$b = (ba')a.$$

We define a'' as $(a')'$, *etc.* Applying identity (1.1) in two ways to $((ba'')a')a$,

(1.2) $$ba'' = ba.$$

This implies $(aa')b' = (aa')b'''$. From (G), $b = (b''a')a$, and from (1.1),

(1.3) $$b = b''.$$

Hence, using (1.1), $(bb')b'' = b = (ba')a''$. From (G), $b' = (a'b')a$. This is equivalent to

(1.4) $$b = (ab)a'.$$

Proceeding as before, $(bb')b'' = b = (ab'')a'$ and from (G), $b' = (aa')b'$ or

(1.5) $$b = (aa')b.$$

Multiplying (1.4) on the right by a and using (1.1), we obtain

(1.6) $$ba = ab.$$

* This paper was written while the author was under contract to the United States Air Force Office of Scientific Research.

Multiplying (1.5) on the right by b' and using (1.1), we obtain $bb'=aa'$. We denote this common value by e. As a consequence of (1.6) and (1.5), we have two theorems.

$$e = xx' = x'x. \tag{1.7}$$

$$x = ex = xe. \tag{1.8}$$

We may now restate (G) in the form: If $b'=(rs')t'$, then $b=(tr')s$. This implies $[(rs')t']'=(tr')s$. Taking primes again, $(rs')t'=[(tr')s]'=(s't')r$. This may be restated as $(ab)c=(ca)b$ and, similarly, $(ca)b=(bc)a$. These identities and (1.6) imply

$$(ab)c = a(bc). \tag{1.9}$$

Thus S is a commutative group and since (G) holds for such groups, (G) characterizes such groups.

2. The inverse operation. Consider a set S closed under an operation, denoted by subtraction, for which we have the identity:

$$y = x - [(x - z) - (y - z)]. \tag{H}$$

We define x^* as $x-x$. Placing $y=x$ in (H), we have

$$x = x - (x - z)^*. \tag{2.1}$$

If $z=(x-y)^*$, it follows from (2.1) that $x-z=x$. And replacing z in (2.1) by this expression gives

$$x = x - x^*. \tag{2.2}$$

Hence, from (H) with $z=x^*$,

$$y = x - [x - (y - x^*)]. \tag{2.3}$$

From this, if $z=x-(y-x^*)$, we have $x-z=y$. Choosing this expression for z in (2.1), we obtain

$$x = x - y^*. \tag{2.4}$$

From this and (2.3) it follows that

$$y = x - (x - y). \tag{2.5}$$

The following theorem is proved by subtracting both differences from x and applying (2.5).

$$\text{If } x - y = x - z, \text{ then } y = z. \tag{2.6}$$

From this and (2.4), we have

$$y^* = z^*. \tag{2.7}$$

We use 0 to denote this common value. We have shown that $z-0=z$. Since $x-y=0$ implies $x=x-(x-y)=y$, we see that

$$x - y = 0 \text{ if and only if } x = y. \tag{2.8}$$

Now consider the expression $x-(x-[(x-z)-(y-z)])$. From (H), it equals $x-y$. From (2.5), it equals $(x-z)-(y-z)$. We have proved

$$x - y = (x - z) - (y - z). \tag{2.9}$$

But (2.5), (2.8), and (2.9) characterize a commutative group in terms of its inverse operation [4]. Thus we have shown that the single identity (H) characterizes commutative groups in terms of the subtraction operation.

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THE PROBABILITY DISTRIBUTION OF THE PRODUCT OF n RANDOM VARIABLES

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In many cases one must be satisfied with approximate results when treating the distributions of n random variables by n-dimensional mapping. However some problems, in particular this one, yield an exact solution by standard techniques employing characteristic functions and contour integration as indicated in this note.

Assume n random variables $x_1, \cdots, x_n$, each distributed with uniform density over an interval of unit length. That is, denoting the density-function of x_k by f_k:

$$f_k(x) = \begin{cases} 1 \text{ for } |x - a_k| \leqq \frac{1}{2}, \\ 0 \text{ elsewhere,} \end{cases} \qquad k = 1, \cdots, n. \tag{1}$$

Putting $y=x_1x_2 \cdots x_n$, we ask for the probability distribution of y under the assumption that the $x_1, \cdots, x_n$ are stochastically independent. Let x_k be positive and therefore $a_k>\frac{1}{2}$, $k=1, \cdots, n$ in (1), so that we can form

$$u = \log y = \log x_1 + \cdots + \log x_n = u_1, \cdots + u_n, \qquad u_k = \log x_k; \tag{2}$$

then we will have the joint distribution of the $u_1, \cdots, u_n$

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$$g(u_1, \cdots, u_n) = \prod_{k=1}^{n} g_k(u_k), \tag{3}$$

where $g_k(u_k) = f_k(e^{u_k})e^{u_k}$, since the $u_1, \cdots, u_n$ are also stochastically independent and $g_k(u_k) = f_k(x_k(u_k))(dx_k/du_k)$.

We now calculate the characteristic function associated with $g_k(u_k)$, obtaining from (1) and (3)

$$\phi_k(t) = \int_{-\infty}^{+\infty} e^{itu} g_k(u)du = \int_{-\infty}^{+\infty} e^{itu} f_k(e^u)e^u du$$

$$= \int_{|e^u - a_k| \leq 1/2} e^{itu} e^u du = \int_{b_k}^{c_k} e^{u(1+it)} du,$$

where $b_k = \log (a_k - \frac{1}{2})$ and $c_k = \log (a_k + \frac{1}{2})$. Thus

$$\phi_k(t) = \frac{1}{1+it}\left\{(a_k + \tfrac{1}{2})^{1+it} - (a_k - \tfrac{1}{2})^{1+it}\right\}. \tag{4}$$

If we denote the characteristic function associated with the unknown density $g(u)$ by $\phi(t)$ then, since it is an n-fold convolution, it follows by well-known statistical methods* that

$$\phi(t) = \prod_{k=1}^{n} \phi_k(t) = \prod_{k=1}^{n} \frac{(a_k + \tfrac{1}{2})^{1+it} - (a_k - \tfrac{1}{2})^{1+it}}{1+it}. \tag{5}$$

From (5) we derive $g(u)$ by the inversion formula $g(u) = (1/2\pi)\int_{-\infty}^{+\infty} e^{-itu}\phi(t)dt$ and with $u = \log y$ the probability density of y as follows:

$$f(y) = g(u(y))\cdot\frac{du}{dy} = g(\log y)\cdot\frac{1}{y} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} y^{-(it+1)}\phi(t)dt$$

$$= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} y^{-z} \times \prod_{k=1}^{n} \frac{A_k^z - B_k^z}{z}\, dz = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} h(z)dz, \tag{6}$$

where $A_k = a_k + \frac{1}{2}$, $B_k = a_k - \frac{1}{2}$ and the path of integration is the straight line through the point $z = 1$ parallel to the imaginary axis in the complex z-plane.

To evaluate the integral in (6) we first observe that the integrand $h(z)$ is a regular function of z in the entire z-plane, so that for any $w > 0$

$$\int_{1-iw}^{1+iw} h(z)dz + \int_{1+iw}^{iw} h(z)dz + \int_{iw}^{-iw} h(z)dz + \int_{-iw}^{1-iw} h(z)dz = 0. \tag{7}$$

We have from (6)

* H. Cramer, Mathematical Methods of Statistics, Princeton, 1946.

$$\left|\int_{1+iw}^{iw} h(z)dz\right| \leqq \frac{1}{w^n} \cdot \operatorname*{Max}_{0\leqq x\leqq 1} y^{-x} \cdot \prod_{k=1}^{n} (A_k^x + B_k^x) \leqq Cw^{-n} \qquad (n \geqq 1)$$

and $Cw^{-n} \to 0$ as $n \to \infty$. Similarly,

$$\left|\int_{-iw}^{1-iw} h(z)dz\right| \leqq Cw^n \to 0 \text{ as } n \to \infty.$$

Therefore from (6) and (7)

$$f(y) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{y^{-z}}{z^n} \prod_{k=1}^{n} (A_k^z - B_k^z)dz, \tag{8}$$

where the path of integration is the imaginary axis. Denoting by $R(w)$ the path $z = we^{i\phi}$, $-\pi/2 \leqq \phi \leqq \pi/2$ we have from (8)

$$f(y) = \lim_{w\to\infty} \frac{1}{2\pi i}\int_{R(w)} y^{-z} \cdot \prod_{k=1}^{n} (A_k^z - B_k^z)\ \frac{dz}{z^n} \cdot \tag{9}$$

Now we express the product in the integrand as a sum of terms, each of which is a product of simple powers with exponent z, thus itself a power with exponent z. Using the abbreviation $P(A) = \prod_{k=1}^{n} A_k$ we get

$$\begin{aligned}\prod_{k=1}^{n} (A_k^z - B_k^z) &= (P(A))^z - \sum_{k=1}^{n}\left(B_k \frac{P(A)}{A_k}\right)^z \\ &\quad + \sum_{k_1,k_2=1,k_1<k_2}^{n}\left(B_{k_1}B_{k_2}\frac{P(A)}{A_{k_1}A_{k_2}}\right)^z \cdots + (-1)^n\left(\prod_{k=1}^{n} B_k\right)^z.\end{aligned} \tag{10}$$

From this we see that the integrand in (9) can be written as a sum of 2^n terms of the special form*

$$\pm \exp\left(-z(\log y - \log C_r(A_1, \cdots, A_n, B_1, \cdots, B_n)\right) = \pm \exp\left(-zc_r(y, A, B)\right),$$

where the c_r in the exponential terms are real and defined by

$$y^{-z}\prod_{k=1}^{n} (A_k^z - B_k^z) = \sum_{r=1}^{2^n} \epsilon_r e^{-zc_r} \tag{11}$$

as functions of the parameters $A_1, \cdots, A_n, B_1, \cdots, B_n, y$; and where $\epsilon_r = \pm 1$ according to whether the number of factors B_k appearing in c_r is even or odd.

Now we have for $c_r \geqq 0$

$$\left|\int_{R(w)} e^{-zc_r}\frac{dz}{z^n}\right| \leqq \pi w w^{-n} \to 0 \qquad (n \geqq 2), \tag{12}$$

as $w \to \infty$. (This can be shown to be 0 for $n = 1$ also.) Thus from (9) and (11)

* We write exp (Z) for e^Z.

$$f(y) = \sum_{\substack{r=1 \\ c_r(y,A,B)<0}}^{2^n} \epsilon_r \cdot \lim_{w\to\infty} \frac{1}{2\pi i} \int_{R(w)} e^{-c_r z} \cdot \frac{dz}{z^n}. \tag{13}$$

Now denoting by $L(w)$ the half circle $z = we^{i\phi}$, $\pi/2 \leqq \phi \leqq 3\pi/2$, so that $R(w) + L(w)$ is the full circle $K(w)$ given by $|z| = w$, we have for $c_r < 0$ similar to (12):

$$\left| \int_{L(w)} e^{-c_r z} \cdot \frac{dz}{z^n} \right| \to 0$$

as $w \to \infty$ for $n \geqq 2$. Therefore for $c_r < 0$

$$\lim_{w\to\infty} \frac{1}{2\pi i} \int_{R(w)} e^{-c_r z} \cdot \frac{dz}{z^n} = \lim_{w\to\infty} \frac{1}{2\pi i} \int_{K(w)} e^{-c_r z} \cdot \frac{dz}{z^n} - \lim_{w\to\infty} \frac{1}{2\pi i} \int_{L(w)} e^{-c_r z} \cdot \frac{dz}{z^n} = \frac{(-c_r)^{n-1}}{(n-1)!}.$$

Finally from (13) for $n \geqq 2$

$$f(y) = \frac{1}{(n-1)!} \sum_{\substack{r=1 \\ c_r(y)<0}}^{2^n} \epsilon_r (-c_r(y))^{n-1} \tag{14}$$

where $c_r(y)$ and ϵ_r can be found from (10) and (11). Computing $f(y)$ formally from (14) also for $n = 1$ we get $f(y) = f(x_1) = f_1(x_1)$ as defined in (1), so that now (14) is valid for all $n \geqq 1$, and gives us the probability density of the random variable $y = \prod_{k=1}^{n} x_k$.

To illustrate the result in (14), we use as an example the case $n = 3$. From (10) and (11) we determine the ϵ_r and c_r as follows:

$$\begin{aligned}
&c_1 = \log y - \log A_1A_2A_3;\ \epsilon_1 = 1; && c_5 = \log y - \log A_3B_1B_2;\ \epsilon_5 = 1;\\
&c_2 = \log y - \log A_2A_3B_1;\ \epsilon_2 = -1; && c_6 = \log y - \log A_2B_1B_3;\ \epsilon_6 = 1;\\
&c_3 = \log y - \log A_1A_3B_2;\ \epsilon_3 = -1; && c_7 = \log y - \log A_1B_2B_3;\ \epsilon_7 = 1;\\
&c_4 = \log y - \log A_1A_2B_3;\ \epsilon_4 = -1; && c_8 = \log y - \log B_1B_2B_3;\ \epsilon_8 = -1.
\end{aligned} \tag{15}$$

Assuming the order relation $\frac{1}{2} < a_1 \leqq a_2 \leqq a_3$, we obtain an ordering also for the various products of the A_k and B_k by the following reasoning: We have $A_k = a_k + \frac{1}{2}$, $B_k = a_k - \frac{1}{2} > 0$. Now the function $f(x) = (x-\frac{1}{2})/(x+\frac{1}{2})$ has the derivative $f'(x) = 1/(x+\frac{1}{2})^2$, which is positive for all x. Therefore $f(x)$ is monotonic increasing with increasing x and it follows that $B_1/A_1 \leqq B_2/A_2 \leqq B_3/A_3$.
Thus

$$B_1B_2B_3 < B_1B_2A_3 \leqq B_1A_2B_3 < B_1A_2A_3,\ A_1B_2B_3 < A_1B_2A_3 \leqq A_1A_2B_3 < A_1A_2A_3.$$

Here we have $B_1A_2B_3 \leqq A_1B_2B_3$ and $B_1A_2A_3 \leqq A_1B_2A_3$, but not necessarily $B_1A_2A_3 = A_1B_2B_3$.

So we have two possible orderings of the eight products. Let us choose one of them by assuming the last order relation also to be valid. Then we get from (14) and (15) by means of the functional equation of the logarithmic function:

$$f(y) = \begin{cases} 0 & (y \leqq B_1B_2B_3), \\ \frac{1}{2}\log^2\frac{y}{B_1B_2B_3} & (B_1B_2B_3 \leqq y \leqq B_1B_2A_3), \\ \frac{1}{2}\log^2\frac{y}{B_1B_2B_3} - \frac{1}{2}\log^2\frac{y}{B_1B_2A_3} & (B_1B_2A_3 \leqq y \leqq B_1A_2B_3), \\ \log\frac{A_2}{B_2}\log\frac{A_3}{B_3} - \frac{1}{2}\log^2\frac{y}{B_1A_2A_3} & (B_1A_2B_3 \leqq y \leqq B_1A_2A_3), \\ \log\frac{A_2}{B_2}\log\frac{A_3}{B_3} & (B_1A_2A_3 \leqq y \leqq A_1B_2B_3), \\ \log\frac{A_2}{B_2}\log\frac{A_3}{B_3} - \frac{1}{2}\log^2\frac{y}{A_1B_2B_3} & (A_1B_2B_3 \leqq y \leqq A_1B_2A_3), \\ \frac{1}{2}\log^2\frac{y}{A_1A_2A_3} - \frac{1}{2}\log^2\frac{y}{A_1A_2B_3} & (A_1B_2A_3 \leqq y \leqq A_1A_2B_3), \\ \frac{1}{2}\log^2\frac{y}{A_1A_2A_3} & (A_1A_2B_3 \leqq y \leqq A_1A_2A_3), \\ 0 & (A_1A_2A_3 \leqq y). \end{cases}$$

FINITE MARKOV CHAINS AND THEIR APPLICATIONS*

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Recent applications of mathematics to the behavioral and biological sciences have emphasized the importance of Markov chains as a stochastic model. Because of this interest, it seemed important to develop a unified treatment of the descriptive quantities of a Markov chain. I shall describe briefly such a treatment developed with John G. Kemeny.†

A finite Markov chain is a special type of stochastic process which may be described briefly as follows. There are a finite number of *states* which we label $1, \cdots, r$. The process moves through these states in a sequence of *steps*. If

* Invited address at the Thirty-ninth Summer Meeting of the Mathematical Association of America, August 25, 1958.

† The details will appear in a forthcoming book, Finite Markov Chains by John G. Kemeny and J. Laurie Snell.

at any time it is in state i, it moves, on the next step, to state j, with probability p_{ij}. The matrix $P=\{p_{ij}\}$ is called the *matrix of transition probabilities*. The process is completely determined by specifying the transition matrix P and a starting state.

The importance of matrix theory to Markov chains comes from the fact that the ijth entry of the nth power P^n of P represents the probability that the process will be in state j after n steps if it is started in state i.

The study of the general Markov chain can be reduced to the study of two special types of chains. These are *absorbing chains* and *ergodic chains*. I shall first discuss absorbing chains.

An absorbing chain is defined as follows. A state is *absorbing* if, once entered, it cannot be left. A chain is an *absorbing chain* if it has at least one absorbing state and if, from every state, it is possible (not necessarily in one step) to reach an absorbing state.

An absorbing chain will, with probability one, eventually reach an absorbing state, no matter how it is started. Once in an absorbing state, it stays there. Hence we are interested in the behavior of the process up to the time it reaches an absorbing state. A quite complete description can be given in terms of certain random variables determined by the process. These are:

u_j: The number of times the process is in the nonabsorbing state j before being absorbed.

v: The number of steps taken before absorption.

w: The number of different nonabsorbing states entered before absorption.

x: The state in which the process is absorbed.

For these random variables, only x has a simple distribution. For the others we content ourselves with finding the first two moments (and hence also the variance). These moments will depend upon the starting state. Let me introduce the following notation:

$Pr_i[p]$: The probability that p occurs when the process is started in state i.

$M_i[f]$: The mean value of the random variable f when the process is started in state i.

$C_i[f, g]$: The covariance of f and g when the process is started in state i.

The mean values of the random variables u_j, v, w, are to be represented by vectors and matrices since they depend on the starting state. For example, the random variables u_j determine a matrix $N=\{M_i[u_j]\}$ while the random variable v determines a vector $\tau=\{M_i[v]\}$.

We begin by putting the transition matrix in a canonical form: we put the absorbing states first and then partition the matrix as follows.

$$P=\begin{array}{c} \\ \text{absorbing} \\ \text{nonabsorbing} \end{array}\overset{\begin{array}{cc}\text{absorbing} & \text{nonabsorbing}\end{array}}{\left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right)}.$$

Here I is an indentity matrix and 0 a matrix with all entries zero.

We can then represent all of our interesting quantities in terms of a single fundamental matrix. This is the matrix $N=(I-Q)^{-1}$. In terms of this matrix we can show that,

$$N = \{M_i[u_j]\} = (I - Q)^{-1}, \qquad N_2 = \{M_i[u_j^2]\} = N(2N_{dg} - I),$$
$$\tau = \{M_i[v]\} = Ne, \qquad \tau_2 = \{M_i[v^2]\} = (2N - I)Ne,$$
$$H = \{M_i[w]\} = NN_{dg}^{-1}e, \qquad B = \{Pr_i[x = j]\} = NR.$$

Here e is a vector with each component 1, and N_{dg} is a matrix with diagonal entries the same as N, but zeros for all other entries. The important fact to notice is that all of this information about the process can be obtained by very simple matrix operations from the single fundamental matrix $N=(I-Q)^{-1}$. Hence only one matrix inversion is necessary. This makes it easy to write a general-purpose machine program to compute all of these quantities for a given absorbing chain. We have prepared such a program and have found it invaluable in studying specific applications.

I shall not verify all of these formulas, but let me indicate the proof of the first assertion. The ijth entry of Q^n represents the probability of being in the nonabsorbing state j after n steps, if the chain is started in the nonabsorbing state i. Let $e_j^{(n)}$ be a random variable which has the value 1, if the chain is in state j after n steps, and 0, if it is not. Then $q_{ij}^{(n)}=M_i[e_j^{(n)}]$ and $u_j=\sum_{j=0}^{\infty} e_j^{(n)}$. From this it follows that $M_i[u_j]=\sum_{n=0}^{\infty} M_i[e_j^{(n)}]=\sum_{n=0}^{\infty} q_{ij}^{(n)}$. But $N=(I-Q)^{-1}$ $=I+Q+Q^2+\cdots$. Hence $N=\{M_i[u_j]\}$. That is, the ijth entry of N represents the mean number of times in state j when the process is started in stage i. The fact that the fundamental matrix itself has such a simple probabilistic interpretation is helpful in finding its relation to other basic quantities. The use of eigenvalues which do not have simple interpretations often leads to unnecessarily complicated formulas.

I shall next discuss ergodic chains, and we shall see that their behavior is quite different from that of absorbing chains. An *ergodic chain* is one such that it is possible to go (not necessarily in one step) from any state to any other state. An important special case of an ergodic chain, called a *regular chain*, is one such that, for some N, P^N has no zero entries.

For an ergodic chain

$$\lim_{n\to\infty} \frac{P + P^2 + \cdots + P^n}{n} = A,$$

where A is a matrix with each row the same vector $\alpha=(a_1, \cdots, a_r)$. The vector α has all components positive and is the unique vector with components adding to 1 and such that $\alpha P=\alpha$. For regular chains this limit may be strengthened to $\lim_{n\to\infty} P^n=A$. In this case we see that the probability $p_{ij}^{(n)}$ of going from i to j in a large number of states is essentially independent of the starting state. In either case, a_j gives the fraction of the times that the process can be expected

to be in each of the states in the long run. Since we expect, in an ergodic chain, that the process will continually wander around through all of the states, we are interested in quite different random variables than we considered for absorbing chains.

Interesting descriptive random variables for an ergodic chain deal, for example, with the number of times a process is in a particular state in the first n steps, and in the behavior of the process between occurrences of two states, or before returning to a starting state for the first time. I shall discuss two such random variables. These are:

v_j: The time required to reach j for the first time.
$s_j(n)$: The number of times in state j in the first n steps.

As in the case of an absorbing chain we are interested in means and variances of these random variables. The quantity $M_i[v_j]$, for $i \neq j$, represents the average length of time, starting at i, to reach j for the first time. It is called the *mean first passage time*. $M_i[v_i]$ represents the mean time to return to state i and is called the *mean recurrence time*. We represent these quantities by the matrix $M = \{M_i[v_j]\}$.

We define first a fundamental matrix Z for regular chains to play the role of N for absorbing chains. This matrix is given by $Z = (I - P + A)^{-1}$.

For the regular case,

$$Z = I + (P - A) + (P^2 - A) + \cdots .$$

In this case the entry z_{ij} of Z may then be interpreted as a measure of the deviations of $p_{ij}^{(n)}$ from their limiting probabilities a_j. From Z we can obtain the following information about $s_j^{(n)}$ and v_j.

$$M = \{M_i[v_j]\} = (I - Z + EZ_{dg})D,$$

$$W = \{M_i[v_j^2]\} = M(2Z_{dg}D - I) + 2(ZM - E(ZM)_{dg}),$$

$$C = \left\{\lim_{n\to\infty} \frac{C_k[s_i^{(n)}, s_j^{(n)}]}{n}\right\} = \{a_i z_{ij} + a_j z_{ji} - a_i\delta_{ij} - a_i a_j\}.$$

(Here D is a diagonal matrix with entry $1/a_j$ as the jth diagonal entry, E is a matrix with all 1's, and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.)

The first two formulas provide us with the mean and variances for the first passage times and the recurrence times. One of the advantages of simple matrix expressions is that they suggest simply relations between basic quantities. For example, it can be shown that the mean first passage times completely determine the transition matrix, and a formula for P in terms of these means can be given.

An important use of the covariance matrix C is the following. The central limit theorem for Markov chains states that $(s_j^{(n)} - na_j)/\sqrt{nc_{ij}}$ is asymptotically normally-distributed with mean 0 and standard deviation 1. Thus the diagonal

entries of C provide us with the proper norming constants to apply the central limit theory.

Finally I should like to mention a few of the areas of recent applications of finite Markov chain theory.

In physics, the Ehrenfest urn model has been used as a simple model to describe diffusion. In this model there are two urns which contain altogether N balls. Each second, one of the N balls is chosen at random and moved from the urn which contains it to the other urn. A Markov chain may be formed in two different ways. The first I shall call the *microscopic chain*. For this chain we assume that the balls are distinguishable and label them $1, \cdots, N$. A state is then an N-component vector which has ith component 1 if the ith ball is in the first urn and 0 otherwise. The second chain, the *macroscopic chain*, is formed by taking as state the number of balls in the first urn. The macroscopic chain is obtained from the microscopic chain by combining states. This is not always a legitimate procedure, but in this case it is. Most interesting questions for applications relate to the macroscopic chain. For example, an interesting quantity is the mean time it takes to go from all balls in one urn to as near an equal number in each urn as possible. It is interesting to compare these with the time required to go from an equal number to one of the extreme cases where all are in one urn. These mean first passage times are most easily obtained from the microscopic chain. This chain is represented as a random walk on an N-dimensional cube and the mean first passage time from state i to state j in the macroscopic chain becomes the mean length of time to go a distance $|i-j|$ in the microscopic chain.

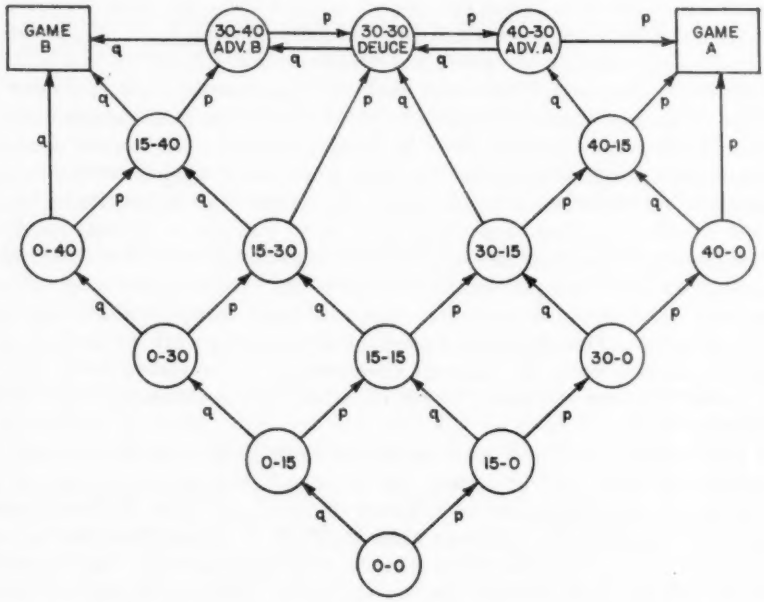
In psychology, the Estes learning model leads to a finite Markov chain. The model has been used to predict the learning habits of animals as well as humans. For example, consider the behavior of rats in a T maze. Different types of feeding schedules lead to different kinds of Markov chains. When the animal is fed always on one side, the model leads to an absorbing chain, and the mean length of time to absorption determines the mean time to complete learning. Other feeding schedules lead to ergodic chains where the number of times in a state related to the number of correct choices the rat makes.

In economics, Markov chain theory has been applied to study the Leontief model for the economy. In this case the transition probability represents the way that a one-dollar order of goods from a single industry spreads its effect through the other industries. Markov chain theory determines, for example, conditions under which economy can meet a given demand.

In genetics, Markov chain models have been used to describe the effect of inbreeding. A pair of offspring are chosen and mated; then a pair of their offspring chosen, *etc.* The state is the genetic type of the parents. An absorbing chain is obtained, and the absorbing states are states where both parents are dominant or both are recessive. The absorption probabilities give the probability that the process will lead to each of these two pure types. The mean time to

absorption gives the mean time to reach a pure type. The number of different types that occur before absorption is also of interest.

Finally, let me mention an application on the lighter side. This is the application of Markov chain theory to the game of tennis. The states and transition probabilities are shown below. (We assume that 30-30 is deuce). The transition probabilities have been determined by assuming that a player has a fixed probability p for a win on each point. (Obvious modifications of this assumption suggest themselves and lead to new Markov chains.)



We have here an absorbing chain. The absorption probabilities give the probabilities of a win, and the mean time to absorption gives the mean length of a game. One would hope that this crude model is not realistic, for it leads to the following predictions.

	$p=.51$	$p=.60$
Probability of winning a point	.510	.600
Probability of winning a game	.525	.736
Probability of winning a set	.573	.966
Probability of winning a match	.635	.9996

One feels that there would be less tennis played between unequal players if this model were correct.

TRIGONOMETRIC SUMS IN ELEMENTARY NUMBER THEORY

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1. Introduction. Let n and r be integers with $r>0$. We say that a complex-valued arithmetical function $f(n)=f(n, r)$ is a *periodic* function of n (mod r) if $f(n+r, r)=f(n, r)$ for all n. The class of periodic functions (mod r) is characterized by the fact that such functions possess a trigonometric (finite Fourier) expansion. For a discussion of the periodic functions (mod r) and their relation to elementary geometry, we mention I. J. Schoenberg [16].

An arithmetical function $f(n, r)$ is said to be an *even* function of n (mod r) if $f((n, r), r)=f(n, r)$ for all n, where (n, r) denotes the greatest common divisor of n and r. Clearly the even functions (mod r) form a subclass of the periodic functions (mod r). This class of functions was studied in a previous paper [2]. The purpose of the present paper is to illustrate the theory of even functions (mod r) with some instructive examples at an elementary level.

First we recall some results proved in [2] which will be needed in our discussion. Ramanujan's trigonometric sum $c(n, r)$ is defined by

$$c(n, r) = \sum_{(x,r)=1} e(nx, r) \qquad (e(n, r) = e^{2\pi in/r}), \tag{1}$$

where the summation is over a reduced residue system (mod r). The function $c(n, r)$ is even (mod r); moreover ([2], Th. 1), a function $f(n, r)$ is even (mod r) if and only if it possesses a trigonometric representation of the form

$$f(n, r) = \sum_{d|r} \alpha(d, r)c(n, d). \tag{2}$$

The Fourier coefficients $\alpha(d, r)$ are determined uniquely by the formula ([2], (8)),

$$\alpha(d, r) = \frac{1}{r\phi(d)} \sum_{a \pmod r} f(a, r)c(a, d), \tag{3}$$

where the summation is over a complete residue system (mod r) and $\phi(r)$ denotes the Euler ϕ-function.

It was also proved in [2] that $f(n, r)$ is even (mod r) if and only if it possesses an arithmetical representation of the form ([2], Th. 3)

$$f(n, r) = \sum_{d|(n,r)} g(d, r/d), \tag{4}$$

where $g(a, b)$ is an arithmetical function of two variables, and the summation is over the common divisors of n and r. It was also shown ([2], (10)) that the Fourier coefficients $\alpha(d, r)$ of $f(n, r)$, as defined in (4), are given by the formula

* The author wishes to express his appreciation to the Institute for Advanced Study, Princeton, N. J., for the use of its facilities in completing this paper, 1957.

$$\alpha(d, r) = \frac{1}{r} \sum_{e \mid (r/d)} g(r/e, e)e. \tag{5}$$

We now give a brief description of the content of this paper. Suppose that k is a positive integer. In Sec. 2 we list a number of results concerned with the Jordan totient function $J_k(r)$ ([11], pp. 95–97]; [7], pp. 147–155), which will be required in the later discussion. In Sec. 3 we discuss a function $c^{(k)}(n, r)$ generalizing $c(n, r)$, and prove two results concerning this function (Th. 1, 2), which yield, in case $k=1$, two fundamental evaluations of $c(n, r)$. It is noted that Theorem 1 was proved in an equivalent form by Carmichael [1] (see the remark in Sec. 2) and later by Daniloff ([6], (67)). The proof given in this paper is along the lines of Ramanujan's proof for the special case $k=1$.

Suppose that s is an arbitrary nonnegative integer. In Sec. 4 we obtain the trigonometric expansion (Th. 3) of the function $\sigma_s^*(n, r)$, defined to be the number of solutions $x_0, y_0, \cdots, x_s, y_s \pmod r$ of the congruence,

$$n \equiv x_0y_0 + \cdots + x_sy_s \pmod r. \tag{6}$$

An arithmetical representation of $\sigma_s^*(n, r)$ is deduced in Theorem 4. For a discussion of an extension of this problem using other methods we refer to [3], Sec. 3 and [4], Sec 3. Denote now the greatest common divisor of integers $a_0, \cdots, a_s$ by $(a_0, \cdots, a_s)$ and let $\phi_s^*(n, r)$ represent the number of solutions $x_0, \cdots, x_s \pmod r$ of the congruence,

$$n \equiv x_0 + \cdots + x_s \pmod r, \tag{7}$$

under the restriction $((x_0, \cdots, x_s), r)=1$. In Theorems 5 and 6 of Sec. 4 we obtain results for $\phi_s^*(n, r)$ analogous to those proved for $\sigma_s^*(n, r)$.

It is of interest to note that the finite expansions (2) of even functions (mod r) correspond to Ramanujan's infinite series expansions [15] for certain types of arithmetical functions of a single variable. In fact, Theorems 3 and 6 provide finite analogues of two of Ramanujan's most familiar expansions. More precisely, let $\mu(r)$ denote the Möbius μ-function and define

$$\text{(a) } \sigma_s(r) = \sum_{d \mid r} d^s, \qquad \text{(b) } \phi_s(r) = \sum_{d \mid r} d^s \mu(r/d), \tag{8}$$

or equivalently,

$$\text{(a) } \frac{\sigma_s(r)}{r^s} = \sum_{d \mid r} \frac{1}{d^s}, \qquad \text{(b) } \frac{\phi_s(r)}{r^s} = \sum_{d \mid r} \frac{\mu(d)}{d^s}. \tag{9}$$

In Sec. 6, by a special limiting process, we deduce Ramanujan's expansion of $\sigma_s(r)/r^s$ from the results of this paper for $\sigma_s^*(n, r)/r^s$ and his expansion of $\phi_s(n)/n^s$ from the corresponding results for $\phi_s^*(n, r)/r^s$ (Th. 7, 8, respectively). Thus, in a certain sense, the congruence problems (6) and (7) may be conceived of as the ultimate arithmetical sources of Ramanujan's expansions.

Further investigations in the theory of even functions (mod r) will form the substance of a paper to be published later.

2. The Jordan function $J_k(r)$. Let k denote an arbitrary positive integer. As in [5] we define a k-vector, $\{a_i\} = \{a_1, \cdots, a_k\}$, to be an ordered set of k integers, and call two k-vectors $\{a_i\}$, $\{b_i\}$ congruent (mod k, r) provided $a_i \equiv b_i \pmod{r}$, $i = 1, \cdots, k$. A *complete* set of residues (mod k, r) is defined to be a set of k-vectors $\{a_i\}$ where the a_i range independently over a complete residue system (mod r). The Jordan function $J_k(r)$ may be defined as the number of elements in a *reduced* residue system (mod k, r); that is, the number of $\{a_i\}$ in a complete residue system (mod k, r) such that $((a_i), r) = 1$, where $(a_i) = (a_1, \cdots, a_k)$, $(0) = 0$. It is also convenient to define $J_0(r) = 1$ or 0 according as $r = 1$ or $r > 1$. Clearly $J_1(r) = \phi(r)$.

In addition to the above notation we define the sum of two k-vectors by $\{a_i + b_i\} = \{a_i\} + \{b_i\}$ and scalar multiples by $c\{a_i\} = \{ca_i\}$ for integers c. We now collect a number of known results relating to $J_k(r)$.

LEMMA 1. *The k-vectors $\{a_i\} = \{rx_i/d\}$ where d ranges over the divisors of r and, for each d, x_i ranges over a reduced residue system* (mod k, d), *form a complete residue system* (mod k, r).

This lemma ([16], Sec. 2) can be proved by the method used in treating the familiar special case $k = 1$ ([9], Sec. 16.2). An immediate consequence of this result is

LEMMA 2.

$$\sum_{d|r} J_k(d) = r^k. \tag{10}$$

By the Möbius inversion formula ([9], Sec. 16.4) there follows the evaluation,

LEMMA 3.

$$J_k(r) = \phi_k(r) \equiv \sum_{d|r} d^k \mu(r/d). \tag{11}$$

For other derivations of Lemma 3 we mention ([5], Th. 2) and the references listed in that paper.

A function $f(r)$ is said to be factorable if $f(1) = 1$ and $f(r_1 r_2) = f(r_1) f(r_2)$ for all positive integers r_1, r_2, such that $(r_1, r_2) = 1$. In view of the factorability of $\mu(r)$, one deduces ([9], Sec. 16.3, Th. 265) from Lemma 3 the factorability of $J_k(r)$.

LEMMA 4.

$$J_k(r_1 r_2) = J_k(r_1) J_k(r_2), \qquad (r_1, r_2) = 1. \tag{12}$$

Lemmas 2, 3, and 4 generalize familiar properties of $\phi(r)$ ([9], Ch. 16). We note that $J_k(r)$ satisfies the more general factorability law ([8], (30)) contained in the following

LEMMA 5. *For all positive* r_1, r_2,

$$J_k(r_1r_2) = J_k(r_1)J_k(r_2)\frac{\delta^k}{J_k(\delta)}, \qquad \delta = (r_1, r_2). \tag{13}$$

We next prove two useful lemmas concerning residue systems (mod k, r). The term *minimal* (complete) *residue system* (mod k, r) is used to denote the complete residue system $\{a_i\}$ such that $0 \leqq a_i < r$. Similarly, a *minimal reduced residue system* is defined to be the reduced system $\{a_i\}$ satisfying $0 \leqq a_i < r$.

LEMMA 6. *If* $(a, b) = 1$, *then a reduced residue system* (mod k, ab) *is generated by the set* $\{ah_i + bh_i'\}$ *where* $\{h_i\}$ *and* $\{h_i'\}$ *range over reduced residue systems* (mod k, b) *and* (mod k, a) *respectively.*

Proof. The elements of the set $\{ah_i + bh_i'\}$ are distinct (mod k, ab); moreover, $((ah_i + bh_i'), ab) = 1$ for each element of the set. Since, by Lemma 4, there are $J_k(ab)$ elements in the set, it must form a reduced residue system (mod ab). This proves Lemma 6.

The preceding lemma generalizes a well-known result in the case $k = 1$ ([9] Sec. 5.5, Th. 61).

LEMMA 7. *A reduced residue system* (mod k, ab) *can be decomposed into* $J_k(ab)/J_k(b)$ *reduced residue systems* (mod k, b).

Proof. Let $\gamma(r)$ denote the core of r; that is, $\gamma(r) = 1$ if $r = 1$ while $\gamma(r)$ is the product of the distinct prime factors of r if $r > 1$. We separate the proof into three cases.

(i) If $(a, b) = 1$, the lemma results from Lemma 6.

(ii) Next, suppose $\gamma(a) \mid b$. Let S denote the set $\{x_i + by_i\}$ where $\{y_i\}$ ranges over a minimal residue system (mod k, a) while x_i ranges over a minimal reduced residue system (mod k, b). It follows that the elements of S are distinct (mod k, ab) and are contained in a minimal reduced residue system T (mod k, ab). Conversely, by the division algorithm, any element of T can be expressed in the form $\{X_i\} = \{r_i + q_ib\}$ where $0 \leqq q_i < a$, $0 \leqq r_i < b$, $((r_i), b) = 1$. Hence $S = T$ and the lemma follows in this case.

(iii) In the general case, place $a = \alpha\beta$ where $\gamma(\beta) \mid b$ and $(\alpha, b) = 1$. Then by (i) there are $J_k(ab)/J_k(\beta b)$ reduced residue systems (mod k, βb) contained in such a system (mod k, ab). By (ii) there are $J_k(\beta b)/J_k(b)$ reduced residue systems (mod k, b) contained in such a system (mod k, βb). Hence, multiplication gives $J_k(ab)/J_k(b)$ systems (mod k, b) contained in a single reduced residue system (mod k, ab) and the proof is complete.

For a proof of the preceding result in the case $k = 1$, we refer to Nagell ([14], p. 24). The following analogue of Lemma 7 follows from definition.

LEMMA 8. *A complete residue system* (mod k, ab) *contains* a^k *complete residue systems* (mod k, b).

3. A generalization of $c(n, r)$. Ramanujan's trigonometric sum $c(n, r)$ can be generalized by placing

$$c^{(k)}(n, r) = \sum_{((x_i), r)=1} e(n(x_1 + \cdots + x_k), r), \tag{14}$$

where $\{x_i\}$ ranges over a reduced residue system (mod k, r). It is observed that $c^{(1)}(n, r) = c(n, r)$. We shall obtain two evaluations of $c^{(k)}(n, r)$; first we state a simple lemma. Let $\xi(n, r)$ be defined by $\xi(n, r) = r$ or 0 according as $r \mid n$ or $r \nmid n$.

LEMMA 9.

$$\eta(n, r) \equiv \sum_{a \pmod r} e(na, r) = \xi(n, r). \tag{15}$$

This result is well-known ([9], Sec. 16.6).

THEOREM 1.

$$c^{(k)}(n, r) = \sum_{d \mid (n, r)} d^k \mu(r/d). \tag{16}$$

Proof. Place

$$\eta^{(k)}(n, r) = \sum_{a_i \pmod{k, r}} e(n(a_1 + \cdots + a_k), r), \tag{17}$$

where the summation is over a complete residue system (mod k, r). Note that $\eta^{(1)}(n, r) = \eta(n, r)$. By (17) and the additivity property $e(a+b, r) = e(a, r)e(b, r)$,

$$\eta^{(k)}(n, r) = \eta^k(n, r) = \xi^k(n, r). \tag{18}$$

Moreover, by Lemma 1 and (17), we have, since $e(na, ab) = e(n, b)$,

$$\eta^{(k)}(n, r) = \sum_{d \mid r} c^{(k)}(n, d). \tag{19}$$

Comparing (18) and (19) we have

$$\sum_{d \mid r} c^{(k)}(n, d) = \xi^k(n, r). \tag{20}$$

Application of the Möbius inversion formula to (20) gives

$$c^{(k)}(n, r) = \sum_{d \mid r} \xi^k(n, d)\mu(r/d) = \sum_{d \mid r, d \mid n} d^k \mu(r/d),$$

and the theorem is proved.

The case $k = 1$ yields Ramanujan's evaluation ([15], (2.7); [9], Sec. 16.6, Th. 271) of $c(n, r)$.

COROLLARY 1.

$$c(n, r) = \sum_{d \mid (n, r)} d\mu(r/d). \tag{21}$$

If n and r are relatively prime, one obtains

COROLLARY 2. *For all* k,

$$c^{(k)}(n, r) = c^{(k)}(1, r) = \mu(r) \qquad (n, r) = 1. \tag{22}$$

Remark. Carmichael ([1], Sec. 4) introduced a function $c_r^{(k)}(n)$, defined by means of Dirichlet series, which was shown to be equivalent to the function appearing on the right of (16). It therefore follows by Theorem 1 that $c_r^{(k)}(n) = c^{(k)}(n, r)$.

We now deduce a second evaluation of $c^{(k)}(n, r)$.

THEOREM 2.

$$c^{(k)}(n, r) = \frac{J_k(r)\mu(m)}{J_k(m)} \qquad \left(m = \frac{r}{(n, r)}\right). \tag{23}$$

Proof. By (14) we may write

$$c^{(k)}(n, r) = \sum_{((x_i), r)=1} e\left(\frac{n}{(n, r)}(x_1 + \cdots + x_k), m\right).$$

But by Lemma 7 and (22) it follows that

$$c^{(k)}(n, r) = \frac{J_k(r)}{J_k(m)} c^{(k)}\left(\frac{n}{(n, r)}, m\right) = \frac{J_k(r)\mu(m)}{J_k(m)}.$$

This completes the proof.

We observe that in case $k=1$, Theorem 2 reduces to the Hölder evaluation ([10], (13)) of $c(n, r)$ contained in

COROLLARY 3.

$$c(n, r) = \frac{\phi(r)\mu(m)}{\phi(m)} \qquad \left(m = \frac{r}{(n, r)}\right). \tag{24}$$

In case $n=0$, one obtains

COROLLARY 4.

$$c^{(k)}(0, r) = J_k(r). \tag{25}$$

This result also follows from Theorem 1 or by the definition of $c^{(k)}(n, r)$. We remark finally that the Hölder relation (24) was actually proved as early as 1907 by Kluyver ([12], p. 410); however, it was Hölder who first recognized the intrinsic interest of this formula.

4. Representations of $\sigma_s^*(n, r)$ and $\phi_s^*(n, r)$. In this section we obtain expansions of the functions $\sigma_s^*(n, r)$ and $\phi_s^*(n, r)$ in the trigonometric form (2) and in the arithmetical form (4). We first consider $\sigma_s^*(n, r)$.

THEOREM 3. *The number of solutions in x_i, y_i (mod r) of the congruence* (6) *is given by*

$$\sigma_s^*(n, r) = r^{2s+1} \sum_{d|r} (c(n, d)/d^{s+1}). \tag{26}$$

Proof. It is evident that $\sigma_s^*(n, r)$ is even (mod r). Therefore, by (2) and (3), $\sigma_s^*(n, r)$ has a trigonometric representation of the form

$$\sigma_s^*(n, r) = \sum_{d|r} \alpha(d, r)c(n, d), \tag{27}$$

$$\alpha(d, r) = \frac{1}{r\phi(d)} \sum_{a \pmod r} \sigma_s^*(a, r)c(a, d). \tag{28}$$

By the definitions of $\sigma_s^*(n, r)$ and $c(n, r)$ we obtain

$$\begin{aligned}\alpha(d, r) &= \frac{1}{r\phi(d)} \sum_{x_i, y_i \pmod{s+1, r}} c(x_0y_0 + \cdots + x_sy_s, d) \\ &= \frac{1}{r\phi(d)} \sum_{(z,d)=1} \sum_{x_i, y_i \pmod{s+1, r}} e(z(x_0y_0 + \cdots + x_sy_s), d),\end{aligned}$$

where, as in (17), $\{x_i\}$, $\{y_i\}$ range over complete residue systems (mod $s+1$, r). By the additivity of $e(a, r)$ and Lemmas 8 and 9, it follows that

$$\begin{aligned}\alpha(d, r) &= \frac{r^{2s+1}}{\phi(d)d^{2s+2}} \sum_{(z,d)=1} \prod_{i=0}^{s} \Big(\sum_{y_i \pmod d} \eta(zy_i, d) \Big) \\ &= \frac{r^{2s+1}}{\phi(d)d^{2s+2}} \sum_{(z,d)=1} \Big(\sum_{y \pmod d} \xi(zy, d) \Big)^{s+1}.\end{aligned}$$

Since $(z, d) = 1$, by definition of $\xi(n, r)$ and $\phi(r)$ one obtains

$$\alpha(d, r) = r^{2s+1}/d^{s+1}. \tag{29}$$

The theorem follows from (27) and (29).

THEOREM 4.

$$\sigma_s^*(n, r) = r^s \sum_{d|(n,r)} dJ_{s+1}(r/d). \tag{30}$$

Proof. Applying (21) to (26) and recalling the definition (8b) of $\phi_s(r)$, it follows that

$$\begin{aligned}\sigma_s^*(n, r) &= r^{2s+1} \sum_{d|r} (1/d^{s+1}) \sum_{e|(n,d)} e\mu(d/e) \\ &= r^{2s+1} \sum_{e|(n,r)} e \sum_{d|r,\ d=\delta e} (\mu(d/e)/d^{s+1}) \\ &= r^s \sum_{e|(n,r)} e \sum_{\delta|(r/e)} (r/(e\delta))^{s+1}\mu(\delta) = r^s \sum_{e|(n,r)} e\phi_{s+1}(r/e).\end{aligned}$$

The theorem follows on applying Lemma 3.

The following corollaries are special cases of Theorem 4 (and also Theorem 3).

COROLLARY 5. *If* $(n, r)=1$, *then the number of solutions of* (6) *is given by*

$$\sigma_s^*(n, r) = r^s J_{s+1}(r). \tag{31}$$

COROLLARY 6. *If* $n\equiv 0 \pmod r$, *then the number of solutions of* (6) *is given by*

$$\sigma_s^*(n, r) = r^s \sum_{d|r} d J_{s+1}(r/d). \tag{32}$$

Before considering the function $\phi_s^*(n, r)$, we sketch an alternative method for proving the two preceding theorems. Theorem 4 can be proved first by a direct arithmetical approach, on the basis of Lemma 8 and the following lemma:

If $((a_i), r)=1$, *then the number of solutions* $B_s(r)$ *of the congruence* $n\equiv a_0y_0 + \cdots + a_sy_s \pmod r$ *in* $y_i \pmod r$ *is* $B_s(r)=r^s$.

It is sufficient to observe that this result is valid for r a prime power, because $B_s(r)$ is a factorable function of r, by the Chinese Remainder Theorem ([9], Sec. 8.1, Th. 121). Theorem 3 can now be proved on applying (4) and (5) to the arithmetical representation (32) and using Lemma 2. The details are left to the reader.

We illustrate this alternative method by applying it to the function $\phi_s^*(n, r)$. First we introduce the notation,

$$J_s(n, r) = J_s((n, r)), \qquad \phi_s(n, r) = \phi_s((n, r)). \tag{33}$$

THEOREM 5. *The number of solutions of the congruence* (7) *in* $x_i \pmod r$ *such that* $((x_0, \cdots, x_s), r)=1$ *is given by*

$$\phi_s^*(n, r) = \left(\frac{r}{(n, r)}\right)^s J_s(n, r). \tag{34}$$

Proof. If $s=0$, then it is obvious that $\phi_0^*(n, r)=J_0(n, r)=1$ or 0 according as $(n, r)=1$ or $\neq 1$. In the remainder of the proof we may therefore suppose that $s>0$. Under this assumption, the function $\phi_s^*(n, r)$ may be redefined as the number of s-vectors $\{x_i\}$, distinct (mod s, r), such that $((n-x_1-\cdots-x_s, x_1, \cdots, x_s), r)=1$. Now a divisor d of r is a divisor of $((n-x_1-\cdots-x_s, x_1, \cdots, x_s), r)$ if and only if d is a divisor of $((n, x_1, \cdots, x_s), r)$. Therefore, since $((n, x_1, \cdots, x_s), r)=(x_1, \cdots, x_s, (n, r))\equiv((x_i), (n, r))$, it follows that $\phi_s^*(n, r)$ represents the number of s-vectors $\{x_i\}$, (mod s, r), such that $((x_i), (n, r))=1$. Applying Lemma 8, we obtain then, by the definition of $J_s(n, r)$ and the later reformulation of $\phi_s^*(n, r)$,

$$\phi_s^*(n, r) = \left(\frac{r}{(n, r)}\right)^s \phi_s^*(n, (n, r)) = \left(\frac{r}{(n, r)}\right)^s J_s(n, r).$$

THEOREM 6.

$$\phi_s^*(n, r) = \frac{J_{s+1}(r)}{r} \sum_{d|r} \left(\frac{\mu(d)}{J_{s+1}(d)} \right) c(n, d). \tag{35}$$

Proof. By (9b) we may rewrite (34) in the form

$$\phi_s^*(n, r) = r^s \sum_{d|(n,r)} \mu(d)/d^s. \tag{36}$$

Applying (4) and (5) to (36) with $g(a, b) = \mu(a)a^{-s}$ we obtain on the basis of (16),

$$\phi_s^*(n, r) = \sum_{d|r} \alpha(d, r)c(n, d), \tag{37}$$

where

$$\begin{aligned} \alpha(d, r) &= (1/r) \sum_{e|(r/d)} \mu(r/e)e^{s+1} \\ &= (1/r) \sum_{e|(r/d,r)} \mu(r/e)e^{s+1} = (1/r)c^{(s+1)}(r/d, r). \end{aligned} \tag{38}$$

Hence by Theorem 2 it follows that

$$\alpha(d, r) = \{J_{s+1}(r)\mu(d)\}/\{rJ_{s+1}(d)\}. \tag{39}$$

The theorem follows by (37) and (39).

We mention the following special cases.

COROLLARY 7. *If* $(n, r) = 1$, *then the number of solutions of* (7) *is* $\phi_s^*(n, r) = r^s$.

COROLLARY 8. *If* $n \equiv 0 \pmod{r}$, *then the number of solutions of* (7) *is* $\phi_s^*(n, r) = J_s(r)$.

It will be observed that the method used in proving Theorems 3 and 4 can also be applied to prove Theorems 5 and 6. More precisely, Theorem 6 can be proved by applying (2) and (3) to $\phi_s^*(n, r)$ and making use of Lemma 7 and (22). Then Theorem 5 can be proved by applying (21) to (35) with the aid of [4], Th. 9 and (13). The reader may supply the details.

5. Series expansions of $\sigma_s(n)/n^s$ and $\phi_s(n)/n^s$. In this section n will denote a positive integer. First we restate the results of the preceding section in a form convenient for use in this section. Combining Theorems 3 and 4 on the basis of Lemma 3, one obtains

$$\frac{\sigma_s^*(n, r)}{r^s} = \sum_{d|(n,r)} d\phi_{s+1}(r/d) = r^{s+1} \sum_{d|r} \frac{c(n, d)}{d^{s+1}}. \tag{40}$$

Similarly, by Theorems 5 and 6 and Lemma 3, we have

$$\frac{\phi_s^*(n, r)}{r^s} = \frac{\phi_s(n, r)}{(n, r)^s} = \frac{\phi_{s+1}(r)}{r^{s+1}} \sum_{d|r} \left(\frac{\mu(d)}{\phi_{s+1}(d)} \right) c(n, d). \tag{41}$$

The principal tool in passing from (40) and (41) to Ramanujan's expansions of $\sigma_s(n)/n^s$ and $\phi_s(n)/n^s$ will be the following lemma.

LEMMA 10. *Let $\lambda(k)$ be a sequence of positive integers, $k=1, 2, \cdots$, such that for all sufficiently large k, $\lambda(k)$ is divisible by $d=1, \cdots, k$. If, in addition, the series $\sum_{m=1}^{\infty} f(m)$ converges absolutely, then*

$$\lim_{k\to\infty} T_k = \sum_{m=1}^{\infty} f(m) \equiv S, \qquad T_k = \sum_{d|\lambda(k)} f(d). \tag{42}$$

Proof. Let S_k denote the kth partial sum of the series in (42) and let R_k denote the remainder after k terms of the series $\sum_{m=1}^{\infty} |f(m)|$. For k sufficiently large, $\lambda(k) > k$ and

$$\begin{aligned} |T_k - S_k| &= \left| \sum_{d|\lambda(k),\, d>k} f(d) \right| \leqq \sum_{d|\lambda(k),\, d>k} |f(d)| \\ &\leqq \sum_{m=k+1}^{\lambda(k)} |f(m)| < \sum_{m=k+1}^{\infty} |f(m)| \equiv R_k. \end{aligned}$$

But by the hypothesis of absolute convergence, $\lim R_k = 0$ as $k\to\infty$; hence $\lim (T_k - S_k) = 0$ or, equivalently, $\lim T_k = S$ ([13], Ch. 3, 4). This completes the proof.

We shall also need some additional facts.

LEMMA 11. *For fixed $n>0$, $c(n, r)$ is bounded as a function of r.*

Proof. By (21)

$$|c(n, r)| \leqq \sum_{d|(n,r)} d \leqq \sum_{d|n} d \equiv \sigma(n).$$

LEMMA 12 ([18], Th. 10). *If $s>0$, then for every $\epsilon>0$,*

$$\lim_{r\to\infty} \frac{\phi_s(r)}{r^{s-\epsilon}} = \infty. \tag{43}$$

A proof of Lemma 12 in the case $s=1$ appears in ([9], Sec. 18.4, Th. 327). The general case can be proved in an analogous manner.

Let $\zeta(t)$ denote the Riemann ζ-function, $\zeta(t) = \sum_{n=1}^{\infty} n^{-t}$, $t>1$. We shall need the well-known fact ([9], Sec. 17.5, Th. 287; [13], p. 446)

$$\zeta^{-1}(t) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^t} \qquad (t > 1). \tag{44}$$

The series in (44) is absolutely convergent.

We are now in a position to derive Ramanujan's expansions ([15], (6.1), (9.6)) of $\sigma_s(n)/n^s$ and $\phi_s(n)/n^s$. (Also see [9], Sec. 17.5, Th. 292 in the case of $\sigma_s(n)/n^s$.)

THEOREM 7. *If* $s>0$, $n>0$, *then*

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1)\sum_{m=1}^{\infty}\frac{c(n,m)}{m^{s+1}}. \tag{45}$$

Proof. By (40) we have, on transforming slightly and applying (9b),

$$\sum_{d|(n,r)}\frac{1}{d^s}\left(\sum_{e|(r/d)}\frac{\mu(e)}{e^{s+1}}\right) = \sum_{d|r}\frac{c(n,d)}{d^{s+1}}. \tag{46}$$

We shall take limits on both sides of (46) as r ranges over the sequence $r_k=k!$, using $L=L(n)$ to denote the resulting limit. Passing to the limit first on the left of (46), one obtains

$$L = \lim_{k\to\infty}\sum_{d|(n,r_k)}\frac{1}{d^s}\left(\sum_{e|(r_k/d)}\frac{\mu(e)}{e^{s+1}}\right) = \sum_{d|n}\frac{1}{d^s}\lim_{k\to\infty}\left(\sum_{e|(r_k/d)}\frac{\mu(e)}{e^{s+1}}\right).$$

Applying Lemma 10, (44), and (9a), it thus follows, for $s>0$, that

$$L = \left(\sum_{d|n}\frac{1}{d^s}\right)\zeta^{-1}(s+1) = \frac{\sigma_s(n)}{n^s}\zeta^{-1}(s+1) \qquad (s>0). \tag{47}$$

Passing now to the limit on the right of (46), it follows by Lemmas 10 and 11 that

$$L = \lim_{k\to\infty}\sum_{d|r_k}\frac{c(n,d)}{d^{s+1}} = \sum_{m=1}^{\infty}\frac{c(n,m)}{m^{s+1}} \qquad (s>0). \tag{48}$$

Theorem 7 results on comparing (47) and (48).

THEOREM 8. *If* $s>0$, $n>0$, *then*

$$\frac{\phi_s(n)}{n^s} = \zeta^{-1}(s+1)\sum_{m=1}^{\infty}\left(\frac{\mu(m)}{\phi_{s+1}(m)}\right)c(n,m). \tag{49}$$

Proof. By (41) and (9b) we have

$$\frac{\phi_s(n,r)}{(n,r)^s} = \left(\sum_{d|r}\frac{\mu(d)}{d^{s+1}}\right)\sum_{d|r}\left(\frac{\mu(d)}{\phi_{s+1}(d)}\right)c(n,d). \tag{50}$$

We again take limits in (50) as r ranges over the sequence $r_k=k!$ Denoting the limit by $L=L(n)$, we obtain on the left,

$$L = \lim_{k\to\infty}\frac{\phi_s(n,r_k)}{(n,r_k)^s} = \frac{\phi_s(n)}{n^s}. \tag{51}$$

By Lemma 10 and (44), we have for $s>0$,

$$\lim_{k\to\infty} \sum_{d|r_k} \frac{\mu(d)}{d^{s+1}} = \zeta^{-1}(s+1). \tag{52}$$

Also, by Lemma 12, the series $\sum_{m=1}^{\infty} \phi_{s+1}^{-1}(m)$ is absolutely convergent for $s>0$. Hence, application of Lemmas 10 and 11 gives, for $s>0$,

$$\lim_{k\to\infty} \sum_{d|r_k} \left(\frac{\mu(d)}{\phi_{s+1}(d)}\right) c(n, d) = \sum_{m=1}^{\infty} \left(\frac{\mu(m)}{\phi_{s+1}(m)}\right) c(n, m). \tag{53}$$

Passing to the limit on the right of (50), we obtain by (52) and (53),

$$L = \zeta^{-1}(s+1) \sum_{m=1}^{\infty} \left(\frac{\mu(m)}{\phi_{s+1}(m)}\right) c(b, m) \qquad (s>0). \tag{54}$$

Comparison of (51) and (54) yields the theorem.

Remarks. We mention that, although s has been assumed integral in this paper, Theorems 7 and 8 are actually valid for all real $s>0$. The preceding proofs remain valid in the more general case, because it can be shown that (46) is true for all s, while (50) is true for all $s \neq -1$. However, it is the case of integral s that is of chief interest, because it is in this case that the identities (46) and (50) have an arithmetical interpretation in terms of the congruences (6) and (7).

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AN ELEMENTARY TREATMENT OF THE IMBEDDING OF A GRAPH IN A SURFACE

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1. Introduction. The object of this paper is to give a simple proof of the following theorem of König ([1], p. 20; [2], p. 198):

Any connected graph may be imbedded in an orientable surface so as to form the vertices and edges of a map.

2. Definitions. By a *graph* we understand a collection of $N_0(\geqq 1)$ vertices, joined in pairs by N_1 $(\geqq 0)$ edges.

It is *connected* if every two vertices, say A and Z, are joined by a chain of consecutively adjacent edges

$$AB, BC, \cdots, XY, YZ.$$

An edge is called an *isthmus* if its removal leaves the graph disconnected. If every edge of a connected graph is an isthmus, the graph is a *tree* (and $N_0 - N_1 = 1$).

A *map* is the decomposition of an unbounded surface into N_2 simply-connected regions by the vertices and edges of a graph. In other words, the complement of the graph on the surface consists of N_2 simply-connected open regions, namely, polygons whose sides are whole edges of the graph. The *characteristic* of the surface is

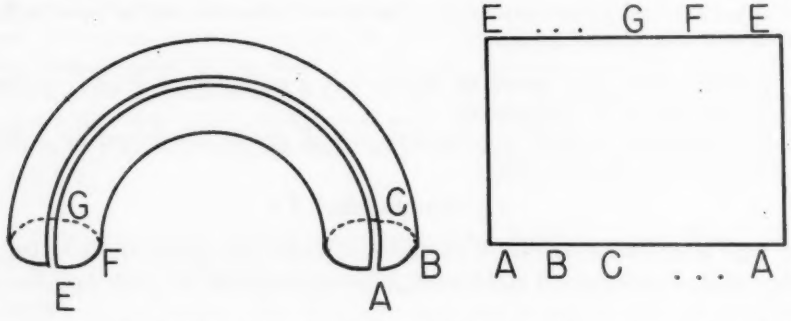
$$\chi = N_0 - N_1 + N_2.$$

If the surface is orientable, its *genus* is

$$p = 1 - \tfrac{1}{2}\chi.$$

(Such a surface may be regarded as a sphere with p handles.) We allow an edge of the map to occur twice among the sides of a region. For instance, the graph that has one vertex and one edge may be imbedded in a sphere to form the map $\{1, 2\}$ whose two regions are monogons. Similarly ([1], p. 101) the graph that has two vertices and one edge may be imbedded in a sphere to form the dual map $\{2, 1\}$ which has only one region, a digon, the sphere being slit by the edge. More generally, any tree can be imbedded in a sphere to form a map whose single region is a $2N_1$-gon.

3. Proof of the theorem. Observing that any graph of one edge may be imbedded in a sphere, we use induction over the number of edges. We assume that every connected graph of N_1-1 edges can be imbedded in some surface. Consider a given connected graph of N_1 edges. If it is a tree, the result is obvious. If not, there must be at least one edge AE that is not an isthmus. By our inductive assumption, we can imbed the rest of the graph in a certain surface to form a map M. If the two vertices A and E both belong to the same region of M, we simply join them on the same surface, thus dissecting that region into two smaller regions. If not, we complete our proof thus. There are two regions $ABC\cdots A$ and $EFG\cdots E$ with one end of AE on the boundary of each. Punch out these two regions and replace them by a tubular handle so as to increase the genus of the surface by 1. This can be done by expanding AE into a thin tube and flaring it at both ends to meet the edges of $ABC\cdots A$ and $EFG\cdots E$.



The modified surface contains a map having the same vertices as M, the same edges plus the extra one AE, and the same regions except that $ABC\cdots A$ and $EFG\cdots E$ (either of which may meet itself along an edge) are replaced by the single region $ABC\cdots AEFG\cdots EA$, which meets itself along the edge AE. Thus the whole graph has been imbedded as desired.

4. Concluding remarks. It may happen that a given graph can be imbedded several distinct ways. For instance, a "figure of eight," having one vertex and two edges, can be imbedded either in a sphere (so that the regions consist of two monogons and a digon) or in a torus (so that there is just one region, a quadrangle). For some graphs, such as the complete 6-point, the simplest imbedding is in a nonorientable surface ([1], p. 116).

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MATHEMATICAL NOTES

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ANOTHER PROOF OF CAUCHY'S GROUP THEOREM

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Since $ab=1$ implies $ba=b(ab)b^{-1}=1$, the identities are symmetrically placed in the group table of a finite group. Each row of a group table contains exactly one identity and thus if the group has even order, there are an even number of identities on the main diagonal. Therefore, $x^2=1$ has an even number of solutions.

Generalizing this observation, we obtain a simple proof of Cauchy's theorem. For another proof see [1].

CAUCHY'S THEOREM. *If the prime p divides the order of a finite group G, then G has kp solutions to the equation $x^p=1$.*

Let G have order n and denote the identity of G by 1. The set

$$S = \{(a_1, \cdots, a_p) \mid a_i \in G,\ a_1a_2 \cdots a_p = 1\}$$

has n^{p-1} members. Define an equivalence relation on S by saying two p-tuples are equivalent if one is a cyclic permutation of the other.

If all components of a p-tuple are equal then its equivalence class contains only one member. Otherwise, if two components of a p-tuple are distinct, there are p members in the equivalence class.

Let r denote the number of solutions to the equation $x^p=1$. Then r equals the number of equivalence classes with only one member. Let s denote the number of equivalence classes with p members. Then $r+sp=n^{p-1}$ and thus $p \mid r$.

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A REMARK ON BOUNDED FUNCTIONS

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Denote by E the class of functions regular and bounded by unity in $|z|<1$. Denote by E^* the subclass of functions of E which are in addition univalent in $|z|<1$. Analogies of various inequalities which are known to hold for functions in the class E have been obtained for functions of the class E^*. For example, it is known [3] that there exist functions in E for which the sequence $\{a_0+\cdots+a_n\}$ $(f(z)=\sum a_nz^n)$ is unbounded. On the other hand, it is shown by Fejér in [1] that if $f\in E^*$ then $|a_0+\cdots+a_n|<1+(1/\sqrt{2})$ for all n.

Hardy [2] has shown that if $f\in E$ then $\lim_{r\to 1}\sqrt{(1-r)}\overline{M}(r)=0$, where $\overline{M}(r)=\sum|a_n|r^n$. This raises the question of the behavior of $\overline{M}(r)$ (as $r\to 1$) for $f\in E^*$. To this end we prove the following theorem

THEOREM. *Let $f(z)=\sum a_n z^n$ map the unit disk upon a region of finite area. Then*

$$\lim_{r\to 1}\overline{M}(r)\left(\log\frac{1}{1-r^2}\right)^{-1/2}=0. \tag{1}$$

Remark. We note that if $f\in E^*$ then f maps $|z|\leqq 1$ upon a finite area. Also if $f(z)$ is merely bounded it need not necessarily map the unit disk upon a finite area.

Proof. Since the unit disk is mapped upon a finite area we have

$$\infty > \iint_{|z|<1}|f'(z)|^2dz=\sum_{n=1}^{\infty}n|a_n|^2.$$

Let $\epsilon>0$ be given. Choose an integer k such that $\sum_{n=k+1}^{\infty}n|a_n|^2<\epsilon$. Now

$$\begin{aligned}\overline{M}(r)&=\sum_{n=0}^{\infty}|a_n|r^n\\&=\sum_{n=0}^{k}|a_n|r^n+\sum_{n=k+1}^{\infty}n^{1/2}|a_n|n^{-1/2}r^n.\end{aligned}$$

Upon applying the Schwarz inequality we have

$$\begin{aligned}\overline{M}(r)&\leqq\sum_{n=0}^{k}|a_n|r^n+\left(\sum_{n=k+1}^{\infty}n|a_n|^2\sum_{n=k+1}^{\infty}n^{-1}r^{2n}\right)^{1/2}\\&\leqq\sum_{n=0}^{k}|a_n|r^n+\left(\sum_{n=k+1}^{\infty}n|a_n|^2\right)^{1/2}\left(\log\frac{1}{1-r^2}\right)^{1/2}.\end{aligned}$$

Hence

$$\limsup_{r\to 1}\overline{M}(r)\left(\log\frac{1}{1-r^2}\right)^{-1/2}\leqq\left(\sum_{n=k+1}^{\infty}n|a_n|^2\right)^{1/2}.$$

But since the left member of the above inequality is independent of k, it follows that (1) holds. This completes the proof.

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GROUPS WHICH INDUCE A PARTITION OF A SET

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A group of transformations of a set can be used in a natural manner to define a partition of the set (*i.e.*, a division of the set into disjoint subsets), by specifying that two elements are in the same subset if there exists a transformation in the group which takes one element into the other.

An abstract group may be realized as a group of transformations in different ways so as to bring about several different partitions of a set. (By realization we mean that the abstract group and the group of transformations are isomorphic, so that distinct members of the abstract group give distinct transformations.) And, conversely, a given partition of a set may sometimes be accomplished by several different abstract groups. If a group G can be realized as a group of transformations of a set in a manner so that it causes a certain partition of the set, then we say that this partition is induced* by G. One method of approach to the problem of determining which groups induce a given partition is to investigate the possible orders for such a group and to limit these possible orders. Some results in this direction are given here.

We first establish an upper limit for these orders. Suppose that a set M is partitioned into n subsets M_i where M_i has m_i elements. If G is a group inducing the partition, then any transformation g in G can be factored into a product of cycles on disjoint letters, and the factorization is unique except for the order of the cycles. Any given cycle must contain only elements of one subset. There are $m_i!$ possible cycles on elements of M_i. Hence there can be at most $m_1! \cdots m_n!$ transformations g in G.

The possible orders for a group inducing a given partition are further limited by the theorem below. We consider the case where M is partitioned into two subsets.

THEOREM. *Let M be divided into subsets M_1 and M_2 having m_1 and m_2 elements, respectively, and let r be the least common multiple of m_1 and m_2. Then the order of any group which induces this partition is a multiple of r.*

Proof. Suppose the subsets are given by $M_1 = [a_1^1, a_2^1, \cdots, a_{m_1}^1]$ and $M_2 = [a_1^2, a_2^2, \cdots, a_{m_2}^2]$, and let G be any group inducing the given partition. Let H be the subgroup of G consisting of all ϕ_i such that $\phi_i(a_1^1) = a_1^1$.

We now examine the index of H in G. For each a_i^1, let π_i be a transformation in G such that $\pi_i(a_1^1) = a_i^1$. These π_i exist since G is transitive on M_1. Now $\pi_1 H = H$, and $\pi_i H \neq \pi_j H$ if $i \neq j$, since

$$\pi_i\phi_u(a_1^1) = \pi_i(a_1^1) = a_i^1 \neq a_j^1 = \pi_j(a_1^1) = \pi_j\phi_v(a_1^1),$$

where ϕ_u and ϕ_v are any transformations in H. Thus H has index at least m_1.

* Note that this usage of the word induce is different from that in which we say that a transformation of a set M induces a transformation in a subset of M.

Now let ρ be any transformation in G. Then $\rho(a_1^1)=a_i^1$ for some a_i^1 in M_1. But

$$\pi_i^{-1}\rho(a_1^1) = \pi_i^{-1}[\rho(a_1^1)] = \pi_i^{-1}(a_i^1) = a_1^1$$

since $\pi_i(a_1^1)=a_i^1$. Thus $\pi_i^{-1}\rho=\phi_j$ for some ϕ_j in H, and $\rho=\pi_i\phi_j$, which is in π_iH. This demonstrates that every transformation in G is in one of these cosets π_iH, $i=1, \cdots, m_1$. Hence H is of index m_1 in G, so that the order of G is divisible by m_1.

Similarly the order of G is divisible by m_2, and therefore by $r=$l.c.m. (m_1, m_2).

The previous theorem generalizes to any finite number of subsets, giving the corollary:

If M is partitioned into subsets $M_1, \cdots, M_n$ where M_i has m_i elements, and if r is the least common multiple of $m_1, \cdots, m_n$, then the order of any group inducing this partition is a multiple of r.

As might be expected, the restriction that a group G inducing a partition must be abelian results in a much smaller number of possible orders for G. The following theorem is concerned with this case.

THEOREM. *Let M be partitioned into n subsets M_i, where M_i has m_i elements, and let G be an abelian group inducing this partition. If k is the order of G, then $k \leqq m_1 \cdots m_n$.*

Proof. Let the subsets be given by $M_i = [a_1^i, a_2^i, \cdots, a_{m_i}^i]$, and let $g = k_1 \cdots k_n$ be a transformation in G, where k_i is a product of cycles on elements of M_i.

Suppose now that two transformations g and g' in G map a given element of M_i, say a_1^i, in the same way. For any element a_j^i of M_i, there exists h in G such that $h(a_1^i)=a_j^i$ since G is transitive on M_i. Thus

$$g(a_j^i) = g[h(a_1^i)] = h[g(a_1^i)] = h[g'(a_1^i)] = g'[h(a_1^i)] = g'(a_j^i),$$

so that g and g' map any element in M_i in the same way. This implies that g and g' have the same factor k_i, since k_i is the only factor which involves elements of M_i. That is, there are at most m_i factors k_i, and therefore no more than $m_1 \cdots m_n$ transformations $g=k_1 \cdots k_n$ in G.

The particular case where M is partitioned into one subset gives the corollary:

If G is an abelian group which is transitive on a set M of m elements, then G has order m.

Proof. From the theorem we have that the order k of G cannot be greater than m. But G is transitive on M so that $k \geqq m$. Hence, $k=m$.

It is known that any group of order m can be written transitively on m

elements. Thus the problem of finding all abelian groups which are transitive on a set of m elements is the same as that of finding all abelian groups of order m.

A special case of the partitioning in the theorem above yields the following result.

THEOREM. *Let M be partitioned into n subsets M_i of m_i elements where the m_i are relatively prime in pairs. If G is an abelian group inducing this partition, then G is a direct product of n subgroups, each of which induces a subset of the partition.*

The complete proof is omitted since it is rather long. Groups H_i of order m_i are constructed so that H_i is transitive on M_i and leaves all other subsets fixed. Then it is shown that these H_i are actually subgroups of G, and that the direct product of these subgroups is G.

In each of the theorems above, groups can be constructed which achieve the indicated bound.

The author wishes to express his appreciation to Professor R. W. Ball for suggesting the problem and for subsequent helpful remarks concerning it.

DESMIC SYSTEMS OF TETRAHEDRONS

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1. Eight lines on a quadric cone. (a) If two tetrahedrons $(T)=DABC$, $(M)=MM'M''M'''$ are harmonic, an edge, say, MM' of (M) meets two opposite edges, say, DA, BC of (T) in two points U', X' which separate the points M, M' harmonically ([1], p. 235). Hence the lines DM, DM' meet the plane ABC on the line AX' in two points S, S' harmonically separated by A, X'; that is, the point S' is an harmonic associate of S for the triangle ABC ([2], p. 246).

Similarly the projections of the vertices M'', M''' of (M) from D upon the plane ABC are the other two harmonic associates of S for the triangle ABC. Thus:

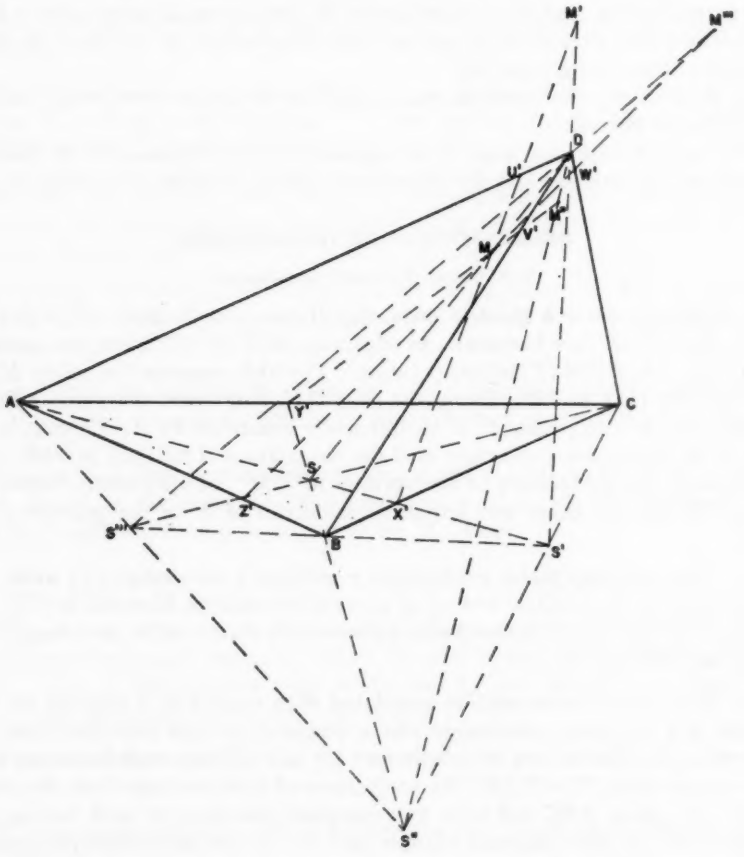
The four lines which project from a vertex of a tetrahedron (T) upon the opposite face the four vertices of a second tetrahedron harmonic to (T), meet that face of (T) in four points harmonically associated for the triangle of that face of (T).

(b) Four points harmonically associated with respect to a triangle are the vertices of a complete quadrangle whose diagonal triangle coincides with the given triangle. Thus if two tetrahedrons (M) and (N) are each harmonic to a given tetrahedron $(T)=DABC$ the projections of their vertices from the point D upon the plane ABC will form two complete quadrangles both having the triangle ABC for their diagonal triangle (art. 1a); hence those eight projections lie on a conic ([3], p. 203). Thus:

If two tetrahedrons are each harmonic to a third tetrahedron, the eight lines joining a vertex of the latter to the vertices of the first two tetrahedrons lie on a cone of the second degree.

2. A skew quartic curve of the first kind through sixteen points. (a) Let (M_i), (N_i) be the tetrahedrons which form a desmic system with the pairs of tetrahedrons (T), (M); (T), (N) (art. 1), respectively ([1], p. 237).

A line joining the vertex D of (T) to a vertex of the tetrahedron (M) passes through a vertex of (M_i) ([1], p. 238, art. 732); hence the vertices of (M_i) lie on the cone (D) having the vertex D of (T) for its vertex (art. 1). The vertices of the tetrahedron (N_i) lie on the cone (D) for analogous reasons. Thus the cone (D) contains all the vertices of the four tetrahedrons (M), (M_i), (N), (N_i).



Moreover, with the vertices A, B, C, of (T) we may associate cones (A), (B), (C), analogous to the cone (D) for the vertex D of (T). Thus the sixteen vertices of the four tetrahedrons lie on four distinct cones. Hence:

If a tetrahedron (T) *belongs to two distinct desmic systems, the sixteen vertices of the remaining four tetrahedrons involved lie on a skew biquadratic curve of the first kind* (C_4).

(b) *The four quadratic cones* (D), (A), (B), (C) *belong to the pencil of quadric surfaces determined by* (C_4), *and the tetrahedron* (T) *is conjugate to all the quadrics of the pencil* ([4], p. 699).

That (T) is conjugate to all the quadrics of the pencil may be shown directly. Any quadric (Q) passing through the skew quartic (C_4) is circumscribed to the tetrahedrons (M) and (M_i) of the desmic system (T), (M), (M_i). If we join a vertex, say, A of (T) to a vertex, say, M of (M), the line AM passes through a vertex, say, M_i of (M_i) and the trace A_0 of the line AMM_i in the face BCD is harmonically separated from A by the two points M, M_i ([1], p. 239, art. 734).

Since the points M, M_i lie on the quadric (Q), the point A_0 is conjugate to A for (Q). Now joining A to the other three vertices of (M) we obtain in the plane BCD three other points analogous to A_0 which proves, superabundantly, that the face BCD of (T) is the polar plane of A for the quadric (Q).

In a like manner it may be shown that the vertices B, C, D, of (T) have for their polar planes with respect to (Q) the faces CDA, DAB, ABC of (T); hence (T) is conjugate to (Q) (cf. [5], p. 307).

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

AN ADJUSTED TRAPEZOIDAL RULE USING FUNCTION VALUES WITHIN THE RANGE OF INTEGRATION*

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The presentation of the trapezoidal rule in textbooks on elementary calculus generally fails to include an estimate of error except in cases where the exact value of the definite integral can be determined by other methods. While such a

* See W. E. Milne, Numerical Calculus, Princeton University Press, 1949, pp. 116–120, for corrections to the trapezoidal rule requiring the knowledge of ordinates *outside* the range of integration.

procedure may tend to give the student an intuitive feeling about the usefulness of the trapezoidal rule, it is hardly a specific guide in situations where the student cannot obtain the exact value of the definite integral or cannot obtain it readily.

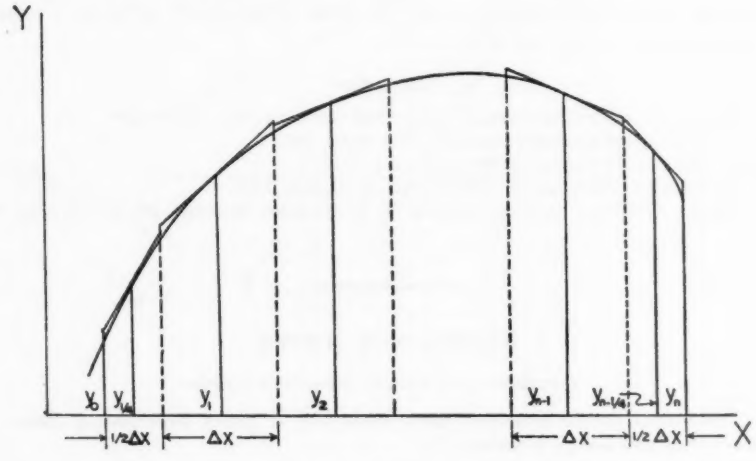
Let us consider the curve $y=f(x)$ in the interval $a \leqq x \leqq b$ throughout which the function is continuous and y'' does not change sign. The exact area under the curve will lie between the value given by the trapezoidal rule

$$K_1 = (\tfrac{1}{2}y_0 + y_1 + \cdots + y_{n-1} + \tfrac{1}{2}y_n)\Delta x \tag{1}$$

and the value given by the tangent trapezoids

$$K_2 = (\tfrac{1}{2}y_{1/4} + y_1 + \cdots + y_{n-1} + \tfrac{1}{2}y_{n-1/4})\Delta x, \tag{2}$$

where $y_{1/4}$ is the ordinate at $x=a+\frac{1}{4}\Delta x$, and $y_{n-1/4}$ is the ordinate at $x=b-\frac{1}{4}\Delta x$. All the ordinates in equation (2) are the medians of the respective tangent trapezoids formed by tangents at the extremities of these ordinates. In addition to y_0 and y_n, the bases of these trapezoids are the perpendiculars to the x-axis erected at the midpoint of each Δx interval.



Let us define $K=\frac{1}{2}(K_1+K_2)$ as the estimate of the area. Then

$$K = [\tfrac{1}{4}(y_0 + y_{1/4} + y_{n-1/4} + y_n) + y_1 + \cdots + y_{n-1}]\Delta x \tag{3}$$

may be called "an adjusted trapezoidal rule."

An upper bound of the error associated with the chosen value of Δx is

$$E = |K - K_1| = |\tfrac{1}{4}([y_{1/4} + y_{n-1/4}) - (y_0 + y_n)]\Delta x| . \tag{4}$$

Thus, by determining only two additional values of the function, namely, at

$x=a+\frac{1}{4}\Delta x$ and at $x=b-\frac{1}{4}\Delta x$, not only is an adjusted approximation obtained for the area but an upper bound of the magnitude of error is provided as well.

Illustrative example. Evaluate $\int_1^5 \ln x dx$ by the trapezoidal rule and by the adjusted trapezoidal rule (3), using $\Delta x=.5$. (The value obtained by integration is 4.0472.)

x	y	Trapezoidal Rule	Adjusted Trapezoidal Rule (3)
1.0	$y_0=$ .00000	.00000	.00000
1.125	$y_{1/4}=$ .11778		.11778
4.875	$y_{n-1/4}=1.58412$		1.58412
5.0	$y_n=1.60944$	1.60944	1.60944
		2)1.60944	4)3.31134
		.80472	.82784
	$y_1+\cdots+y_{n-1}$	7.25665	7.25665
	Sum to be multiplied by Δx	8.06137	8.08449
Estimate of definite integral		4.0307	4.0422
Upper bound of error		No estimate	.0116 (by (4))
Actual error		.0165	.0050

DIFFERENTIATION OF THE REPEATED INTEGRAL

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Most texts on analysis include the rule for differentiating the single integral, namely if

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)dx, \tag{1}$$

then under suitable conditions

$$\frac{d\phi}{d\alpha} = \int_a^b f_\alpha(x, \alpha)dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}. \tag{2}$$

Differentiation of the repeated integral is usually omitted. This author found it interesting and instructive to carry out for the case where the limits of integration are functions of a parameter.

If $f(x, y, \alpha)$ and its partial derivative $f_\alpha(x, y, \alpha)$ are continuous in the region $x_1(\alpha)\leqq x\leqq x_2(\alpha)$, $y_1(x, \alpha)\leqq y\leqq y_2(x, \alpha)$ and x_1, x_2, y_1, y_2 are differentiable functions of α within the region $c\leqq\alpha\leqq d$, then if

$$\phi(\alpha) = \int_{x_1(\alpha)}^{x_2(\alpha)} \int_{y_1(x,\alpha)}^{y_2(x,\alpha)} f(x, y, \alpha)dydx, \tag{3}$$

$$\begin{aligned}\frac{d\phi}{d\alpha} = &\int_{x_1(\alpha)}^{x_2(\alpha)} \int_{y_1(x,\alpha)}^{y_2(x,\alpha)} \frac{\partial f}{\partial \alpha}\, dydx + \int_{x_1(\alpha)}^{x_2(\alpha)} f(x, y_2, \alpha) \frac{\partial y_2}{\partial \alpha}\, dx \\ &- \int_{x_1(\alpha)}^{x_2(\alpha)} f(x, y_1, \alpha) \frac{\partial y_1}{\partial \alpha}\, dx + \int_{y_1(x_2,\alpha)}^{y_2(x_2,\alpha)} f(x_2, y, \alpha) \frac{\partial x_2}{\partial \alpha}\, dy \\ &- \int_{y_1(x_1,\alpha)}^{y_2(x_1,\alpha)} f(x_1, y, \alpha) \frac{\partial x_1}{\partial \alpha}\, dy.\end{aligned} \tag{4}$$

To show this let

$$\psi(x, \alpha) = \int_{y_1(x,\alpha)}^{y_2(x,\alpha)} f(x, y, \alpha)dy. \tag{5}$$

Now using (5), (3) may be written as

$$\phi(\alpha) = \int_{x_1(\alpha)}^{x_2(\alpha)} \psi(x, \alpha)dx,$$

$$\begin{aligned}\phi(\alpha + \Delta\alpha) = &\int_{x_1+\Delta x_1}^{x_2+\Delta x_2} \psi(x, \alpha + \Delta\alpha)dx = \int_{x_1+\Delta x_1}^{x_1} \psi(x, \alpha + \Delta\alpha)dx \\ &+ \int_{x_1}^{x_2} \psi(x, \alpha + \Delta\alpha)dx + \int_{x_2}^{x_2+\Delta x_2} \psi(x, \alpha + \Delta\alpha)dx.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\Delta\phi}{\Delta\alpha} = &\int_{x_1}^{x_2} \frac{\psi(x, \alpha + \Delta\alpha) - \psi(x, \alpha)}{\Delta\alpha}\, dx + \frac{\Delta x_2}{\Delta\alpha} \psi(\epsilon_2, \alpha + \Delta\alpha) \\ &- \frac{\Delta x_1}{\Delta\alpha} \psi(\epsilon_1, \alpha + \Delta\alpha),\end{aligned} \tag{6}$$

where $|x_1 - \epsilon_1| < |\Delta x_1|$ and $|x_2 - \epsilon_2| < |\Delta x_2|$.

The last two terms of (6) follow from the fact that

$$\int_a^b f(x)dx = (b - a)f(\xi), \qquad a < \xi < b.$$

Now as $\Delta\alpha \to 0$, $\epsilon_1 \to x_1$ and $\epsilon_2 \to x_2$. Thus

$$\frac{d\phi}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial \psi}{\partial \alpha}\, dx + \psi(x_2, \alpha) \frac{\partial x_2}{\partial \alpha} - \psi(x_1, \alpha) \frac{\partial x_1}{\partial \alpha}. \tag{7}$$

If we use (5) in (7) we arrive at

$$\frac{d\phi}{d\alpha} = \int_{x_1(\alpha)}^{x_2(\alpha)} \frac{\partial}{\partial\alpha}\left[\int_{y_1(x,\alpha)}^{y_2(x,\alpha)} f(x, y, \alpha)dy\right]dx + \int_{y_1(x_2,\alpha)}^{y_2(x_2,\alpha)} f(x_2, y, \alpha)\frac{\partial x_2}{\partial\alpha}dy$$

$$(8) \qquad - \int_{y_1(x_1,\alpha)}^{y_2(x_1,\alpha)} f(x_1, y, \alpha)\frac{\partial x_1}{\partial\alpha}dy.$$

Using (2) in the first term of (8) we have the result (4).

It follows that under similar conditions this method of repeated application of (2) can be extended to include multiple integrals of any order.

NOTES ON THE FIBONACCI SEQUENCE

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The Fibonacci sequence $\{f_n\}$ is defined by $f_1=f_2=1$, $f_n=f_{n-1}+f_{n-2}$, $n\geqq 3$. Simson noted that

$$(1) \qquad f_{n-1}f_{n+1} - f_n^2 = (-1)^n.$$

This can be proved by induction, for it is obviously true for $n=2$. If (1) is assumed true for $n=k$, then

$$(-1)^k = f_{k-1}f_{k+1} - f_k^2 = f_{k+1}(f_{k-1} + f_k) - f_k(f_k + f_{k+1})$$
$$= f_{k+1}f_{k+1} - f_kf_{k+2} = -(f_kf_{k+2} - f_{k+1}^2),$$

which proves (1) for $n=k+1$.

We can also prove by induction two well-known relations [(2) and (3) below].

$$(2) \qquad f_{n+1} = {}_nC_0 + {}_{n-1}C_1 + {}_{n-2}C_2 + \cdots.$$

Proof. The relation (2) is obviously true for $n=1$ and 2. If it is assumed true for $n=k-1$ and $n=k$, then

$$f_{k+1} = {}_kC_0 + {}_{k-1}C_1 + {}_{k-2}C_2 + \cdots, \qquad f_k = {}_{k-1}C_0 + {}_{k-2}C_1 + \cdots.$$

Adding the two equations we have

$$f_{k+1} + f_k = {}_kC_0 + ({}_{k-1}C_0 + {}_{k-1}C_1) + ({}_{k-2}C_1 + {}_{k-2}C_2) + \cdots.$$

Since ${}_kC_0={}_{k+1}C_0$, ${}_{k-1}C_0+{}_{k-1}C_1={}_kC_1$, *etc.*, then

$$f_{k+2} = {}_{k+1}C_0 + {}_kC_1 + {}_{k-1}C_2 + \cdots,$$

which proves (2) for $n=k+1$.

$$(3) \qquad f_1 + \cdots + f_n = f_{n+2} - 1.$$

Proof. The relation (3) is obviously true for $n=1$. If it is assumed true for $n=k$, then $f_{k+2}-1=f_1+\cdots+f_k$. Adding f_{k+1} to both sides of this equation, we have

$$f_{k+2} + f_{k+1} - 1 = f_1 + \cdots + f_k + f_{k+1}.$$

Hence $f_1 + \cdots + f_{k+1} = f_{k+3} - 1$, which proves (3) for $n = k+1$.

Two other relations [(4) and (5) below] can now be proved.

$$f_{n-2}f_{n+2} - f_n^2 = (-1)^{n+1}. \tag{4}$$

The proof of (4) can be made to depend on (1) as follows:

$$\begin{aligned} f_{n-2}f_{n+2} - f_n^2 &= f_{n-2}(f_{n+1} + f_n) - f_n(f_{n-2} + f_{n-1}) \\ &= f_{n-2}f_{n+1} - f_nf_{n-1} = f_{n-2}(f_{n-1} + f_n) - (f_{n-2} + f_{n-1})f_{n-1} \\ &= f_{n-2}f_n - f_{n-1}^2 = f_{n-2}f_n + f_{n-1}f_n - f_{n-1}f_n - f_{n-1}^2 \\ &= f_n(f_{n-1} + f_{n-2}) - f_{n-1}(f_n + f_{n-1}) = f_n^2 - f_{n-1}f_{n+1} = -(-1)^n, \end{aligned}$$

which proves (4). Adding (1) and (4), we have

$$f_{n-2}f_{n+2} + f_{n-1}f_{n+1} = 2f_n^2 \quad \text{or} \quad f_n = \{(f_{n-2}f_{n+2} + f_{n-1}f_{n+1})/2\}^{1/2}.$$

Also, subtracting (4) from (1), we have

$$f_{n-1}f_{n+1} - f_{n-2}f_{n+2} = 2(-1)^n. \tag{5}$$

ROLLING POLYGONS

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Our purpose here is to obtain perimeters and areas of members of the family of cycloids by rolling polygons upon polygons. Ideas involved consist of elementary plane geometry, some trigonometry, and a speaking acquaintance with the limit concept. The ordinary cycloid, the nephroid, and the astroid serve as examples.

Each demonstration makes use of the trigonometric identities:

$$\sum_{k=1}^{n-1} \sin kx = \frac{\sin \frac{n}{2}x \sin \frac{n-1}{2}x}{\sin \frac{x}{2}}, \tag{I}$$

$$\sum_{k=1}^{n-1} \cos kx = \frac{\cos \frac{n}{2}x \sin \frac{n-1}{2}x}{\sin \frac{x}{2}}. \tag{II}$$

1. The cycloid.* A regular n-gon with circumradius a "rolls" without slipping upon a line (Fig. 1). A vertex P describes a set of $(n-1)$ circular arcs with

* See Problem E 1269, this MONTHLY, vol. 65, 1958, p. 45.

diagonals r_k of the polygon (measured from P) as radii. These radii are $r_k = 2a \sin k(\pi/n)$ and the angle through which the polygon turns at each of its vertices is $2\pi/n$.

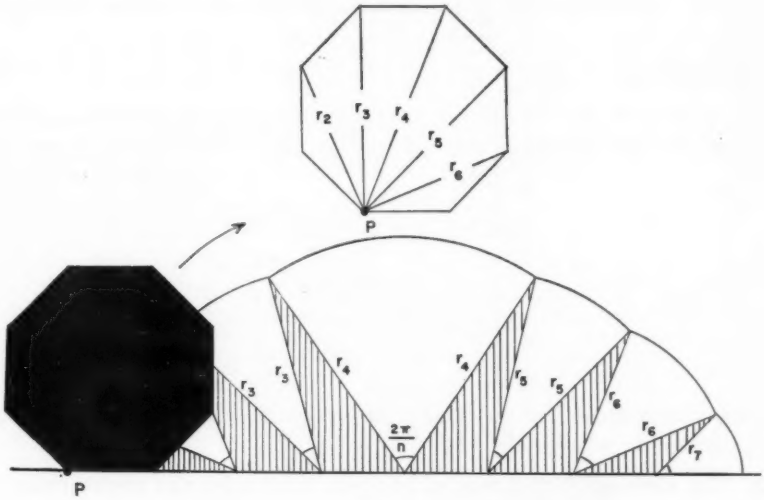

FIG. 1

The sum L_n of arc lengths through one revolution of the polygon is

$$L_n = \frac{2\pi}{n}\sum_{k=1}^{n-1} r_k = \frac{2\pi}{n}\sum_{k=1}^{n-1} 2a \sin k\frac{\pi}{n}$$

$$= \frac{4\pi a}{n}\,\frac{\sin\dfrac{n}{2}\dfrac{\pi}{n}\cdot\sin\dfrac{n-1}{2}\dfrac{\pi}{n}}{\sin\dfrac{\pi}{2n}} \qquad \text{(by (I))}$$

$$= \frac{4\pi a}{n}\,\frac{\sin\left(1-\dfrac{1}{n}\right)\dfrac{\pi}{2}}{\sin\dfrac{\pi}{2n}} = 4\pi a\,\frac{\cos\dfrac{\pi}{2n}}{\left(\sin\dfrac{\pi}{2n}\right)\dfrac{2n}{\pi}}\cdot\frac{2}{\pi}.$$

As $n\to\infty$, the polygon approaches the circle of radius a and $L_n\to 8a$, the length of one arch of the cycloid.

The area between the line and one arch is the sum A_n of areas of the circular sectors plus the sum of the triangular areas whose sides are the diagonals r_k. The latter sum is the area of the polygon itself. Thus

$$A_n = \frac{1}{2}\,\frac{2\pi}{n}\sum_{k=1}^{n-1} r_k^{\,2} = \frac{4\pi a^2}{n}\sum_{k=1}^{n-1}\sin^2 k\,\frac{\pi}{n}$$

$$= \frac{4\pi a^2}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2} - \frac{1}{2}\cos k\,\frac{2\pi}{n}\right)$$

or, by (II),

$$= \frac{2\pi a^2}{n}\left[n - 1 - \frac{\cos\dfrac{n}{2}\,\dfrac{2\pi}{n}\sin\dfrac{n-1}{2}\,\dfrac{2\pi}{n}}{\sin\dfrac{\pi}{n}}\right]$$

$$= \frac{2\pi a^2}{n}\left[n - 1 + \frac{\sin\left(1-\dfrac{1}{n}\right)\pi}{\sin\dfrac{\pi}{n}}\right] = 2\pi a^2,$$

which is independent of n and thus faintly surprising. The area under one arch of the cycloid is accordingly $3\pi a^2$.

2. The nephroid. Let a regular polygon of $2n$ sides and circumradius $a/2$ roll externally on another of circumradius a (Fig. 2). Along one side pivot angles are π/n but where vertex meets vertex the angle of turn is $2\pi/n$. A vertex P of the rolling polygon describes the closed curve composed of circular arcs.

The sum L_n of arc lengths of the closed curve is (since r_{2k} has the double pivot angle)

$$L_n = 2\,\frac{\pi}{n}\left[\sum_{k=1}^{2n-1} r_k + \sum_{k=1}^{n-1} r_{2k}\right]$$

and, since $r_k = a\sin k(\pi/2n)$,

$$L_n = \frac{2\pi a}{n}\left[\sum_{k=1}^{2n-1}\sin k\,\frac{\pi}{2n} + \sum_{k=1}^{n-1}\sin k\,\frac{\pi}{n}\right]$$

$$= \frac{2\pi a}{n}\left[\frac{\sin\dfrac{\pi}{2}\sin\dfrac{2n-1}{2}\,\dfrac{\pi}{2n}}{\sin\dfrac{\pi}{4n}} + \frac{\sin\dfrac{\pi}{2}\sin\dfrac{n-1}{2}\,\dfrac{\pi}{n}}{\sin\dfrac{\pi}{2n}}\right] \qquad \text{(by I)}$$

$$= 2\pi a\left[\frac{4}{\pi}\,\frac{\sin\left(1-\dfrac{1}{2n}\right)\dfrac{\pi}{2}}{\left(\sin\dfrac{\pi}{4n}\right)\dfrac{4n}{\pi}} + \frac{2}{\pi}\,\frac{\sin\left(1-\dfrac{1}{n}\right)\dfrac{\pi}{2}}{\left(\sin\dfrac{\pi}{2n}\right)\dfrac{2n}{\pi}}\right].$$

Thus as $n\to\infty$, $L_n\to 12a$, the perimeter of the nephroid.

The area enclosed by the path of P is the sum A_n of the circular sectors plus the collection of triangular pieces (the latter evidently being double the area of the rolling polygon) plus the area of the fixed polygon. We have

$$A_n = 2\left(\frac{1}{2}\,\frac{\pi}{n}\right)\left[\sum_{k=1}^{2n-1} r_k^2 + \sum_{k=1}^{n-1} r_{2k}^2\right] = \frac{\pi a^2}{n}\left[\sum_{k=1}^{2n-1} \sin^2 k\frac{\pi}{2n} + \sum_{k=1}^{n-1} \sin^2 k\frac{\pi}{n}\right].$$

Replacing $\sin^2\theta$ by $\frac{1}{2}-\frac{1}{2}\cos 2\theta$ and using (II), this may be reduced in the previous manner to $A_n=\pi a^2[(3/2)+(1/n)]$. Thus $A_n\to 3\pi a^2/2$ and the area of the nephroid is accordingly

$$\frac{3\pi a^2}{2} + 2\left(\pi\,\frac{a^2}{4}\right) + \pi a^2 = 3\pi a^2.$$

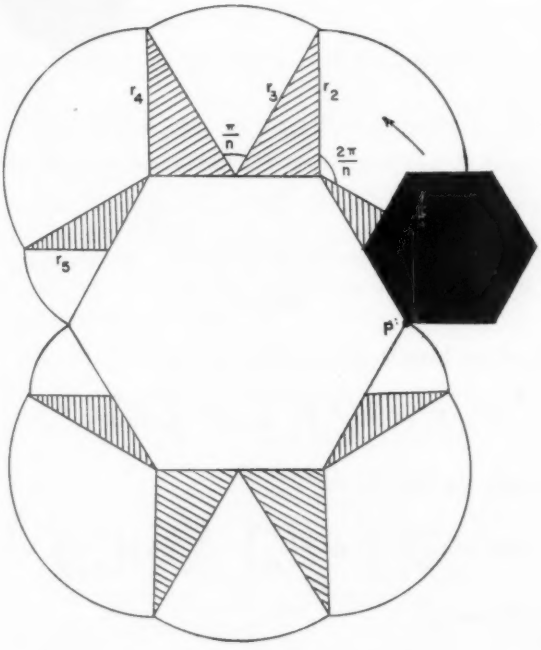

FIG. 2

3. The astroid. Let a regular polygon of $4n$ sides and circumradius $a/4$ roll internally to a similar fixed polygon with circumradius a (Fig. 3). The small polygon rolls through angles $\pi/4n$ at points where its vertices meet the sides of the fixed polygon, but where vertex meets vertex there is no turn. Thus, in

the summation for arc length and area to follow, we deduct contributions of every fourth radial length r_{4k}.

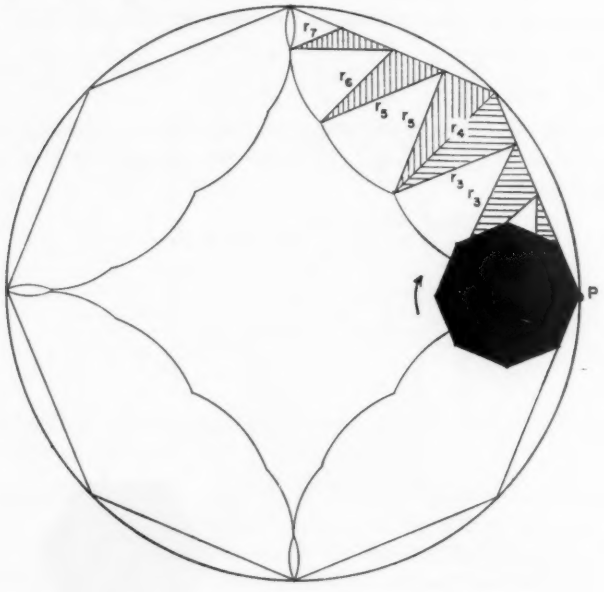

FIG. 3

The sum L_n of arc lengths for a complete trip is

$$L_n = 4\left[\left(\frac{\pi}{4n}\right)\left(\sum_{k=1}^{4n-1} r_k - \sum_{k=1}^{n-1} r_{4k}\right)\right],$$

where $r_k = (a/2)\sin k(\pi/4n)$. Then

$$L_n = \frac{\pi a}{n}\left[\sum_{k=1}^{4n-1} \sin k\left(\frac{\pi}{4n}\right) - \sum_{k=1}^{n-1} \sin k\left(\frac{\pi}{n}\right)\right].$$

Using (I), this reduces to

$$L_n = \pi a\left[\frac{\cos\dfrac{\pi}{8n}}{\dfrac{8n}{\pi}\sin\dfrac{\pi}{8n}}\cdot\frac{8}{\pi} - \frac{\cos\dfrac{\pi}{2n}}{\dfrac{2n}{\pi}\sin\dfrac{\pi}{2n}}\cdot\frac{2}{\pi}\right].$$

As $n\to\infty$, $L_n\to 6a$, the perimeter of the astroid.

For the area, let A_n represent the sum of the circular sectors. Again, r_{4k} creates nothing. Thus

$$A_n = 4\left(\frac{a^2}{4}\right)\left(\frac{\pi}{4n}\right)\left[\sum_{k=1}^{4n-1} r_k^2 - \sum_{k=1}^{n-1} r_{4k}^2\right].$$

Setting

$$r_k^2 = \frac{a^2}{4}\sin^2 k\cdot\frac{\pi}{4n} = \frac{a^2}{8}\left(1 - \cos k\cdot\frac{\pi}{2n}\right)$$

and using (II), $A_n = 3\pi a^2/8$.

The sum of the areas of the triangular portions for each quarter arc is that of the rolling polygon $(na^2/8)\sin(\pi/2n)$, and for the entire trip $(na^2/2)\sin(\pi/2n)$. The fixed polygon has area $2na^2\sin(\pi/2n)$. The area interior to the path of P is then

$$\left(2n\cdot\sin\frac{\pi}{2n}\right)a^2 - \left(\frac{n}{2}\cdot\sin\frac{\pi}{2n}\right)a^2 - \frac{3\pi a^2}{8}.$$

As $n\to\infty$, this has the limit $3\pi a^2/8$, the area of the astroid.

ON PLANE AREA IN POLAR COORDINATES

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Let $r=g(\theta)$ be the polar equation of a single valued, continuous curve, whose equation in rectangular coordinates is $y=f(x)$. As usual, with the pole at the origin of rectangular coordinates, we have $x=r\cos\theta$, $y=r\sin\theta$. Let $P_1(x_1, y_1)$ be a fixed point and $P(x, y)$ a variable point on $y=f(x)$; let $[r_1, \theta_1]$ and $[r, \theta]$ be their polar coordinates, respectively.

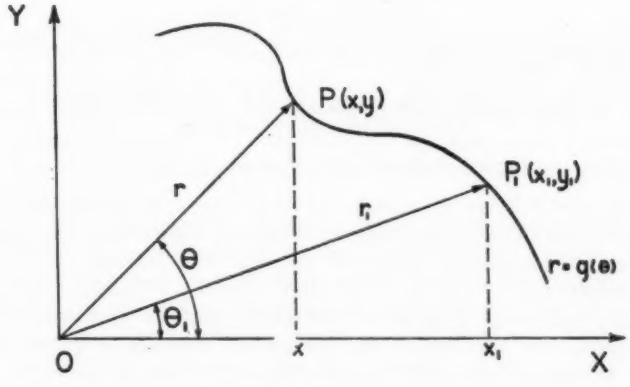

FIG. 1

Referring to Figure 1, let $A(\theta)$ be the area of the variable sector bounded by the curve $g(\theta)$ and the two radius vectors, r and r_1, *i.e.*, OP_1PO. Then

$$A(\theta) = \tfrac{1}{2}xy + \int_{x}^{x_1} f(x)dx - \tfrac{1}{2}x_1y_1. \tag{1}$$

Noting that x_1 and y_1 are constants, we find, differentiating both sides of (1) with respect to θ, that

$$\begin{aligned}\frac{d}{d\theta}A(\theta) &= \frac{1}{2}\left[x\frac{dy}{d\theta} + y\frac{dx}{d\theta}\right] - y\frac{dx}{d\theta} = \frac{1}{2}\left[x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right] \\ &= \frac{x^2}{2}\frac{d}{d\theta}\left(\frac{y}{x}\right) = \frac{r^2\cos^2\theta}{2}\frac{d}{d\theta}(\tan\theta) = \frac{r^2}{2}.\end{aligned} \tag{2}$$

If $[r_2, \theta_2]$ specifies the point P_2 on $g(\theta)$, then it follows that

$$A(\theta_2) = \frac{1}{2}\cdot\int_{\theta_1}^{\theta_2} r^2d\theta = \frac{1}{2}\cdot\int_{\theta_1}^{\theta_2} g^2(\theta)d\theta. \tag{3}$$

The above proof of (3) does not require knowledge of the formula, $\frac{1}{2}r^2\theta$, for the area of a circular sector; and unlike most orthodox proofs in textbooks, application of geometric intuition by the student is obviated, since the fundamental theorem of the integral calculus (*i.e.*, a limiting process involving a sum of areas of circular sectors) is not used. However, a simultaneous presentation of the fundamental theorem and the above proof of (3) would be of some academic value.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND JOHN R. MAYOR, AAAS and University of Maryland

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

A HIGH SPEED COMPUTER COURSE FOR HIGH SCHOOL STUDENTS AND TEACHERS

RICHARD V. ANDREE, The University of Oklahoma

Since the distribution of schools is rather sparse in Oklahoma, it has not seemed feasible to carry out a program of visiting high school lecturers without outside financial assistance. However, last year the University of Oklahoma, with the cooperation of the National High School and Junior College Mathematics Club, undertook a series of lectures in modern mathematics for high school students and teachers at the University campus in Norman. The response was so enthusiastic, with groups coming 200 miles to spend the day, that it was decided to extend the series this year to a full-semester (noncredit) course in *Programming the IBM* 650 *Computer and an Introduction to the Related Mathematics.*

No charge is made for the course, but participants provide their own transportation, luncheon, and text. A grant from the Oklahoma Frontiers of Science Foundation provides reimbursement for the instructor and two laboratory assistants. The University of Oklahoma furnishes classroom space and machine time.

A one-page mimeographed announcement was sent to major high schools in the area in mid-September announcing the course beginning October 4. Applications were accepted only for teacher-student combinations: no teachers without students; no students without a teacher. Classes meet both in the morning (laboratory) and in the afternoon (lecture) on alternate Saturdays during the entire semester. The announcement made it quite plain that a great deal of homework would be demanded of the participants, and that not everyone who wanted to come could be accepted. *Almost three hundred applications were received!* After some hurried consultations and changes in plans, it was decided to enlarge the group to 90 persons. Teacher-participants were chosen, and each teacher permitted to choose his own accompanying student. (In most cases only one student could be accepted per teacher, since there were so many applicants.) It proved difficult to say "no" to students eager enough to commute almost 600 miles (round trip) starting the day before, to take the course, so the class finally made up to 106 persons.

Enthusiasm is running high, and the resulting programs are well written—often ingenious. The experiment of having both teachers and students in the same class is proving highly successful. Each must "save face" by doing excellent work. The homework output is prodigious.

RESEARCH IN EDUCATION: A REPORT ON TWO CONFERENCES

R. M. WHALEY, Executive Director, Advisory Board on Education, NAS-NRC

The improvement of scientific and mathematical education is a primary concern of the Advisory Board on Education of the National Academy of Sciences-National Research Council. In working toward this objective through its various programs, the Board has sensed a need for more knowledge of the fundamental processes of learning. The increasing use of technological aids in education and of experimental programs involving changes of curriculum have emphasized this need.

A group of psychologists queried by the Board agreed that research on intellectual development and on learning in the schools has been seriously neglected. The ABE accordingly has sponsored two conferences of psychologists and educators to study the problem of research in education and recommend solutions.

Dr. Lyle Lanier, Head of the Department of Psychology at the University of Illinois, served as Chairman of the first meeting on April 24–26, 1958, at Easton, Maryland. The conference concluded that: (1) There has not been sufficient research on basic learning processes, possibly due to inadequate financial support; (2) Educational research results have not been systematically or-

ganized, codified, nor interpreted to be accessible to educators; and (3) Research on *human learning* and its application to education should be increasingly stressed.

The Easton Conference recommended consideration of the following: (1) A federal agency concerned with educational research, similar to the National Institute of Mental Health; (2) A nongovernmental agency, perhaps patterned after the National Academy of Sciences-National Research Council or the Social Science Research Council; (3) A summer institute on the psychology of learning.

A follow-up conference on July 9–11 in Madison, Wisconsin, under the chairmanship of Dr. T. R. McConnell, Director of the University of California Center for the Study of Higher Education, was requested to make recommendations for specific action. It recommended the establishment of an Organization for Research in Education to (1) Define the problems fundamental to improvement of education that might be solved by research; (2) Collect and classify existing data; (3) Promote and conduct research on the methodology of learning and of teaching; and (4) Assist in the recruitment and development of trained research personnel.

During the period when the proposed Organization for Research in Education is under consideration, the conference suggested that task force and conference groups be assembled to work on high priority education problems—problems which might be identified by national committees of scientists and mathematicians. Plans should be made this winter for a summer study in 1959.

THE HIGH SCHOOL MATHEMATICS CONTEST*

HENRY L. ALDER, UNIVERSITY OF CALIFORNIA, DAVIS

The eighty-minute contest was given on March 27, 1958, during the first two class periods. As in the past, the test was a multiple-choice examination with five choices listed for each question. The problems covered high school algebra and geometry, but not trigonometry or advanced algebra, as such. The contest examination was not confined to a specific syllabus nor to mere reproduction of classroom work.

Part I of the examination, consisting of 20 questions counting 2 credits each, tested the basic skills and techniques ordinarily associated with plane geometry and elementary and intermediate algebra. In Parts II (20 questions counting 3 credits each) and III (10 questions counting 5 credits each), however, the questions probed beyond such reproduction into the student's ability to meet new situations. The attempt was made, directly and indirectly, to convey to the students some of the fascination of mathematics, some of its broader vistas.

To discourage guessing, students were penalized for incorrect answers. The score S is calculated according to the formula $S=C-\frac{1}{4}(T-C)$, where C is the

* An official report on the contest is on page 171 of this issue.

number of points correct, T the number of points attempted.

For the purpose of awards, the United States and Canada were divided into nine regions. Every school sending in its results had a winner and received at least one award. To attempt to set up grade norms, or to standardize these tests in other ways, would have been basically contradictory to the objectives of the National Committee. However, to permit some comparable judgment of performance, the national median school-team score (obtained by adding the three highest individual scores), the median score for the first place winner, and lower and upper quartiles were published in a *Summary of Results and Awards*, which was sent to each participating school. The top three individuals in the 1958 contest were Michael Day, University High School, Urbana, Illinois (146.25 points), Jason E. Grosz, Bronx High School of Science, Bronx, New York (143.75 points), and Robert E. Kibler, Deering High School, Portland, Maine (136.3 points). The top three school-teams were from Bronx High School of Science (371.75 points), Brooklyn Technical High School (363.00 points), and Abraham Lincoln High School of Brooklyn (355.25 points).

For each region a bronze cup was awarded to the school with the highest team score. An engraved Certificate of Merit, engrossed with the name of the school and signed by the local contest chairman, was awarded on a regional basis to each of the schools with scores in the first decile of all participating schools, exclusive of the bronze-cup winner. One hundred books of mathematical tables, donated by the Chemical Rubber Publishing Company of Cleveland, Ohio, were distributed equitably to the highest-ranking students in each region. A gold pin with a facsimile of the seal of the Mathematical Association of America was awarded to the highest scorer in each participating school making a report.

While the National Committee does not make any awards other than those mentioned above, some of the local sections have built up an extensive system of local awards, consisting of scholarships, U. S. Saving Bonds, merchandise, *etc.*, which are financed by local business. Some sections used the contest as a screening device for participation in a second-stage test which is proctored at various centers. Thus, for example, the "Central Valley's Mathematics Quiz" in Northern California selected from the national contest the five top-scoring high school students in each county of that region to compete in a contest proctored at five centers. The winner of that contest, 15 year old John I. Castor of Clovis, California, received a $1000 Scholarship donated by the McClatchy newspaper chain.

The National Committee's encouragement for local administration of the contest to insure better attention to local problems has resulted in establishment of local Contest Committees in all but three of the Association's twenty-seven sections. Several large sections have organized several committees in accordance with geographical subdivisions of their regions. There are at present 32 local Contest Committees in the United States and Canada.

A MATHEMATICS-PHYSICS COURSE FOR IN-SERVICE TEACHERS

The Electric Boat Division of the General Dynamics Corporation, Groton, Connecticut, offered an intensive, integrated course in mathematics and physics for teachers at the secondary level in the summer of 1958, and is offering a second course this year.

Course description follows:

Calculus, Newtonian Mechanics, and Topics in Modern Physics

An intensive course in the mathematics necessary to the understanding of Newtonian mechanics and its descendants. The course emphasized ideas rather than techniques, but skill in differentiation and integration was suggested as a prerequisite.

Finite Mathematics, The Mathematics of Social Science

Topics include elements of logic, sets, and subsets, permutations and combinations, probability and statistics, vector and matrix algebra with applications in the social and physical sciences.

INDUSTRY EMPLOYS A SPECIAL TEACHER FOR A HIGH SCHOOL

In 1956 Olin Mathieson's Packaging Division of Monroe, Louisiana, undertook a plan to help qualify more students in the local high schools for careers in the physical sciences. The company underwrites the service of one outstanding teacher each year at one of the local high schools. The teacher is chosen to offer a course in mathematics, physics, or chemistry at a level more advanced than the normal curriculum can provide. It was believed that this plan would stimulate both the student body and the faculty alike.

An advanced chemistry class was organized at Neville High School in Monroe for the 1957–58 school session. A physics teacher has been employed for this year. The plan is made possible by a $10,000 grant used primarily to cover the teacher's salary.

JUNIOR-YEAR ENROLLEES IN SCIENCE AND MATHEMATICS

A report* of the Office of Education reveals that in November 1957 there were approximately 50,500 (from 1,104 four-year colleges and universities) junior-year enrollees in science and mathematics as a major field of study. These students are about 12.9% of all junior-year students. About 80.9% of the total number majoring in fields of science and mathematics were men.

Based on the above total enrollment figure, it is estimated that between 40 and 45 thousand bachelor's degrees in science and mathematics will be completed in the 1958–59 school year. There were 33,800 bachelor's degrees in the same fields in 1956–57.

REPORT OF THE COMMISSION ON MATHEMATICS

The report is expected to be ready for distribution about March 31. The December 1958 date (this MONTHLY, vol. 65, 1958, p. 773) was an error.

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* Junior-Year Science and Mathematics Students by Major Field of Study, Circular 520, prepared by the U. S. Department of Health, Education and Welfare, Office of Education. U. S. Government Printing Office, Washington 25, D. C. 56 pp. $0.45.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1351. *Proposed by J. F. Darling, Woodstown, New Jersey*

Find the locus of the centers of the equilateral triangles whose sides a, b, c pass through three fixed points A, B, C respectively.

E 1352. *Proposed by C. W. Trigg, Los Angeles City College*

In setting the type for the multiplication $(abc)(bca)(cab)=234235286$, $a>b>c$, in which the unit's digit is 6, the remaining digits of the product became pied. Restore them to their proper order.

E 1353. *Proposed by L. A. Kenna, University of Arizona*

What is the eccentricity of an ellipse formed by a plane cutting the axis of a right circular cylinder at an angle θ?

E 1354. *Proposed by P. L. Chessin, University of Maryland*

If $(1+x)^{p-2}=1+a_1x+a_2x^2+\cdots+a_{p-2}x^{p-2}$, where p is a prime, then a_1+2, a_2-3, a_3+4, $\cdots$ are all multiples of p.

E 1355. *Proposed by A. J. Goldman, National Bureau of Standards*

If $y(x)>0$ and $y=\log\,[1+(\log x)/x+y/x]$ for all sufficiently large x, prove that $y\sim(\log x)/x$ as $x\to\infty$. (This is a step whose details are left to the reader in a paper by S. Chowla and F. C. Auluck, *Some properties of a function considered by Ramanujan*, J. Indian Math. Soc., vol. 4, 1940, pp. 169–173.)

SOLUTIONS

A Gathering of Six People

E 1321 [1958, 446]. *Proposed by C. W. Bostwick, Riverdale, Maryland*

Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other.

I. *Solution by John Rainwater, University of Washington.* Consider a fixed person A. Of the other five, there are either three whom A knows or three whom he does not. In the first instance, the three are either complete strangers, or two of them are acquaintances and form a trio of such with A. The other case is similar.

II. *Solution by J. D. Baum, Oberlin College.* If the six people are identified

with the vertices of an octahedron, and if each edge and diagonal is colored red or blue according as the two people it connects are strangers or acquaintances, then the problem reduces to Problem 2, Part I, William Lowell Putnam Prize Examination, 1953 [this MONTHLY, vol. 60, 1953, p. 541).

Also solved by W. E. F. Appuhn, Philip Bacon, D A. Breault, E. W. Brown, G. C. Bush, Fitch Cheney, R. J. Cormier, R. J. Driscoll, G. W. Erwin, Jr., Susan L. Friedman, Fred Galvin, L. D. Goldberg, Michael Goldberg, R. E. Greenwood, Cornelius Groenewoud, Werner Held, R. L. Helmbold, J. H. Hodges, R. Holt, A. R. Hyde, Irving Katz, J. D. E. Konhauser, William Kruskal, Joe Lipman, T. M. Little, D. B. Lloyd, R. F. McDermot, J. H. McKay, E. W. Marchand, D. C. B. Marsh, Helen Marston, J. B. Muskat, M. J. Pascual, R. I. Purry, Gustave Rabson, C. A. Reiher, Moses Richardson, J. T. Rosenbaum, Azriel Rosenfeld, Jack Silver, S. E. Spielberg, R. H. Sprague, and the proposer. Late solutions by L. D. Goldstone, F. W. Luttmann, and D. L. Silverman.

The Putnam problem may also be found in G. Gamov and M. Stern, *Puzzle-Math*, New York, 1958, pp. 93–95. This problem has been generalized by R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math., vol. 7, 1955, pp. 1–7.

Several solvers showed that 6 is the smallest number with the stated property. Galvin showed that in any group of 18 people, there are four who are mutually acquainted or unacquainted, and that 18 is the smallest number with this property. Cheney stated that in any gathering of 6 people, there is a second trio who are either mutually acquainted or unacquainted.

Modified Harmonic Series

E 1322 [1958, 446]. *Proposed by J. W. Andrushkiw, Seton Hall University*

Multiply the first p terms of the harmonic series by $(-1)^k$, the next q terms by $(-1)^{k+1}$, the next p terms by $(-1)^k$, the next q terms by $(-1)^{k+1}$, and so on, alternately, thus forming a series denoted by $H(p, q, k)$. Show that $H(p, q, k)$ is convergent if and only if $p=q$.

I. *Solution by R. H. Sprague, University of Kentucky.* (a) Obviously $\sum_{n=r+1}^{r+p} 1/n > \sum_{n=r'+1}^{r'+p} 1/n$ whenever $r'>r$.

(b) Thus $H(p, p, k)$ acts like an alternating series whose terms approach zero, and thus converges.

(c) If $q>p$, omit the last $q-p$ terms from each q-block of terms in $H(p, q, k)$, forming a new series $H'(p, q, k)$. Because of (a), $H'(p, q, k)$ converges. The omitted terms are all of the same sign; hence we may assume them to be positive. Then each term in the nth q-block is no less than the last term in that block, $1/n(p+q)$; thus the sum of the $q-p$ omitted terms is no less than $(q-p)/n(p+q)$. But $\sum_{n=0}^{\infty} (q-p)/n(p+q)$ clearly diverges; hence $H(p, q, k)$ diverges.

(d) Finally, if $p>q$, we note that the absolute value of the last term in the nth p-block is $1/[np+(n-1)q] > 1/n(p+q)$ and proceed as in (c).

II. *Solution by A. E. Danese, Union College.* If $p=q$, the series is of the form $\sum a_n b_n$ with $\sum a_n$ having bounded partial sums and the sequence $\{b_n\}$ monotonically decreasing to zero. Convergence follows by Dirichlet's test [Knopp, *Theory and Application of Infinite Series*, London, 1928, p. 315].

If $p \neq q$, divergence follows immediately by the theorem due to Cesàro [*ibid.*, p. 318, 4]: If $\sum a_n$ converges, but not absolutely, $|a_n|$ is monotonically decreasing, p_n is the number of positive terms and q_n is the number of negative

terms a_m for $m \leqq n$, then $\lim_{n\to\infty} p_n/q_n = 1$, if the limit exists.

The problem occurs as an exercise [*ibid.*, p. 150, 52].

Also solved by Fred Galvin, L. D. Goldberg, Vern Hoggatt, R. H. Hou, J. H. Jordan, Joe Lipman, D. C. B. Marsh, Douglas Maurer, D. L. Muench, M. J. Pascual, Azriel Rosenfeld, Jack Silver, W. A. Veech, Clement Winston, and the proposer. Late solution by J. B. Muskat.

Concerning the Roots of a Special Polynomial

E 1323 [1958, 446]. *Proposed by Harry Goheen, Oregon State College*

Prove that all roots but one of the equation

$$nx^n = 1 + x + x^2 + \cdots + x^{n-1}$$

have absolute value less than 1.

I. *Solution by Joe Lipman, University of Toronto.* When $|x| \geqq 1$,

$$\begin{aligned} |1 + x + x^2 + \cdots + x^{n-1}| &\leqq |1| + |x| + |x|^2 + \cdots + |x|^{n-1} \\ &\leqq n|x|^n = |nx^n|, \end{aligned}$$

with equality holding throughout if and only if $x = 1$. Thus, if

$$1 + x + x^2 + \cdots + x^{n-1} = nx^n,$$

$|x|$ must be <1, except for the one-fold root $x = 1$.

II. *Solution by J. B. Roberts, Reed College.* Putting $x = 1/y$ yields

$$\begin{aligned} nx^n - x^{n-1} - x^{n-2} - \cdots - x - 1 \\ = (1/y - 1)(1/y^{n-1})(y^{n-1} + 2y^{n-2} + 3y^{n-3} + \cdots + n). \end{aligned}$$

The zeros of the last factor on the right are in absolute value >1 by a theorem of Kakeya: "*If* $0 < a_0 < a_1 < \cdots < a_n$, *then all the roots of*

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

are outside the unit circle" (see Tôhoku Math. J., vol. 2, 1912, pp. 140–142, or Dienes, *The Taylor Series*, Ex. 17, p. 67). Thus the zeros of the left side (other than 1) are in absolute value <1.

III. *Solution by A. E. Danese, Union College.* Since $x = 1$ is clearly a root of the equation, it remains to show that the zeros of

$$f(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$$

have absolute value <1.

A. The zeros of $f(x)$ lie in the annulus $1/2 \leqq |x| \leqq 1 - 1/n$, since if all the coefficients of the polynomial

$$p_0z^n + p_1z^{n-1} + \cdots + p_{n-1}z + p_n$$

are positive, its zeros lie in $\alpha \leqq |z| \leqq \beta$, where α is the smallest and β the largest of

$$p_1/p_0,\ p_2/p_1,\ p_3/p_2,\ \cdots,\ p_n/p_{n-1}.$$

(Pólya and Szegö, *Aufgaben und Lehrsätz aus der Analysis*, Erster Band, New York, 1945, p. 88, 23.)

B. Or, $f(x)$ is the derivative of $1+x+x^2+\cdots+x^n$, whose zeros are the nth roots of unity. The result follows with the application of the theorem (*ibid.*, p. 89, 31): The zeros of the derivative of the polynomial $P(z)$, which are not zeros of $P(z)$, lie in the interior of the smallest convex polygon which contains the zeros of $P(z)$.

Also solved by David Adorno and Joel Owen (jointly), R. G. Albert, W. E. F. Appuhn, Merrill Barnebey, D. R. Barr, D. A. Breault, B. H. Brown, A. G. Clark, Eleanor G. Dawley, E. L. Ellis, Fred Galvin, L. D. Goldberg, Michael Goldberg, A. G. Grace, Jr., Norman Greenspan, Emil Grosswald, Vern Hoggatt, R. H. Hou, Irving Katz, Jesse Krehbiel, D. C. B. Marsh, R. A. Miller, M. J. Pascual, John Rainwater, Azriel Rosenfeld, Jack Silver, R. H. Sprague, W. A. Veech, David Zeitlin, and the proposer. Late solutions by R. G. Albert, David Fink, and L. I. Lowell.

Brown pointed out that a generalization of this problem has found application in the sociological sciences. See E. D. Domar, *Depreciation, replacement and growth—and fluctuation*, Economic J., vol. 47, 1957, pp. 654–658. Also see R. M. Solow, *A note on dynamic multipliers*, Econometrica, vol. 19, 1951, pp. 306–16.

Tic-tac-toe as a Game of Chance

E 1324 [1958, 447]. *Proposed by F. E. Clark, Rutgers University*

Suppose tic-tac-toe is turned into a game of pure chance as follows. Designate the squares by 1, $\cdots$, 9, numbering them from left to right by successive rows, starting with the top row. Place a set of nine chips, labeled 1, $\cdots$, 9, in a bag. The first player, A, draws a chip at random and enters an X in the corresponding square. The second player, B, draws at random from the remaining chips and enters an O in the corresponding square. The game ends, of course, as soon as three like entries are obtained which lie along a line (horizontally, vertically, or diagonally). Find the probability that A will win, draw, or lose the game. Also find the probability, for each drawing, that the game will end when that chip is drawn.

Solution by T. M. Little, University of California. The number of combinations of 5 X's and 4 O's in the nine squares is $9!/5!4!=126$, since any combination of 5 X's determines the four O's and vice versa. These 126 combinations can be classified into five groups as follows: (1) 22 combinations containing 2 winning combinations for X; (2) 40 containing a single winning combination of X's; (3) 12 containing a single winning combination of O's; (4) 36 containing both a winning combination of X's and one of O's; and (5) 16 containing no winning combinations for O or X, thus resulting in a draw after the eighth chip is drawn.

For class (1) the probability is 1/5 that A will win with the fifth chip, 3/5 that A will win with the seventh chip, and 1/5 that he will win with the ninth chip. For class (2) the probability is 1/10 that A will win with the fifth chip, 3/10 that he will win with the seventh chip, and 3/5 that he will win with the

ninth. For class (3) the probability is 1/4 that B will win with the sixth chip and 3/4 that he will win with the eighth. For class (4) the probability is 1/10 that A will win with the fifth chip, 9/40 that B will win with the sixth, 9/40 that A will win with the seventh, and 9/20 that B will win with the eighth. For class (5) it is certain that the game will end in a draw when the eighth chip is drawn.

Multiplying these probabilities by the frequency of the corresponding class, we get the following probabilities:

A winning with 5th chip:
$(22/126)(1/5)+(40/126)(1/10)+(36/126)(1/10)=120/1260$
B winning with 6th chip:
$(12/126)(1/4)+(36/126)(9/40)=111/1260$
A winning with 7th chip:
$(22/126)(3/5)+(40/126)(3/10)+(36/126)(9/40)=333/1260$
B winning with 8th chip:
$(12/126)(3/4)+(36/126)(9/20)=252/1260$
A winning with 9th chip:
$(22/126)(1/5)+(40/126)(3/5)=284/1260$
Game ending in draw:
$16/126=160/1260$

A's total probability of winning is therefore 737/1260 and of losing is 363/1260.

Also solved by Fred Galvin, Michael Goldberg, Werner Held, A. R. Hyde, Joe Lipman, D. C. B. Marsh, J. A. O'Brien, Azriel Rosenfeld, and the proposer.

Not all these solutions agreed with the above.

The Matrix Equation $X^2-2AX+B=0$

E 1325 [1958, 447]. *Proposed by Peter Treuenfels, Brookhaven National Laboratory, Upton, L. I., New York*

Let A, B, X denote $n\times n$ matrices. Show that a sufficient condition for the existence of at least one solution X of the matrix equation

$$X^2 - 2AX + B = 0$$

is that the eigenvalues of the $2n\times 2n$ matrix

$$R = \begin{bmatrix} A & I \\ A^2 - B & A \end{bmatrix}$$

be pairwise distinct. Here I denotes the $n\times n$ identity matrix.

Solution by the proposer. Since the eigenvalues of R are pairwise distinct, there exists a $2n\times 2n$ matrix S which diagonalizes R; thus $S^{-1}RS=M$, where M is the diagonal matrix of the eigenvalues of R. Let

$$S = \begin{bmatrix} T & U \\ V & W \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix},$$

where T, U, V, W, L, N denote $n \times n$ matrices and where L and N are diagonal matrices. Our proof will be in two parts; we shall show that: (1) by permuting the eigenvalues of R, if necessary, we can always achieve that T is nonsingular; and (2) that $X = TLT^{-1}$ is a solution of the given matrix equation.

(1) The transforming matrix S is nonsingular; hence its $2n$ row-vectors are linearly independent, whence, *a fortiori*, its first n row-vectors are linearly independent. Hence the rank of the $n \times 2n$ submatrix $[TU]$ of S is n, and at least one $n \times n$ submatrix T^0 of $[TU]$ exists which is nonsingular. By shifting columns of the original transforming matrix we can bring this nonsingular submatrix T^0 into the upper left-hand corner of the transforming matrix. Let us denote the transforming matrix thus obtained by S^0. $(S^0)^{-1}$ is obtained from S^{-1} by a shift of the corresponding rows. Let $M^0 = S^0R^0(S^0)^{-1}$; then M^0 is obtained from M by a shifting of corresponding rows and columns. But, since M is diagonal, this is equivalent to a permutation of the eigenvalues of R. We may therefore assume that the eigenvalues of R have been ordered in such a way that T is nonsingular.

(2) Since $RS = SM$, or

$$\begin{bmatrix} A & I \\ A^2 - B & A \end{bmatrix}\begin{bmatrix} T & U \\ V & W \end{bmatrix} = \begin{bmatrix} T & U \\ V & W \end{bmatrix}\begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix},$$

we obtain $AT + V = TL$, $A^2T - BT + AV = VL$. Solving the first of these equations for V, and substituting for V in the second equation, we find $TL^2 - 2ATL + BT = 0$. Multiplying by T^{-1} on the right yields $TL^2T^{-1} - 2ATLT^{-1} + B = 0$, or $X^2 - 2AX + B = 0$, for $X = TLT^{-1}$.

Also solved by Jack Silver.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4828. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Do the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ converge, where

$$a_{n+1} = \int_0^1 \min(x, b_n, c_n)dx,$$

$$b_{n+1} = \int_0^1 \text{mid } (x, c_n, a_n)dx, \qquad c_{n+1} = \int_0^1 \max (x, a_n, b_n)dx,$$

and mid $(a, b, c) = b$ if $a \geqq b \geqq c$.

4829. *Proposed by J. de Groot, University of Amsterdam, The Netherlands*

Give a simple example of a continuum P such that, for every countable group G, there exists a group of topological transformations of P onto itself which is isomorphic to G.

4830. *Proposed by D. B. Larson, Cornell Aeronautical Laboratory, Buffalo, N. Y.*

Evaluate the integral

$$\int_{-1}^{1} e^{iAz}(1 - Bx^2)^{-1}(1 - x^2)^{-1/2}dx,$$

where A and B are positive, real constants, $B < 1$ and $i^2 = -1$.

4831. *Proposed by J. L. Massera and J. J. Schäffer, Institute of Mathematics and Statistics, Montevideo, Uruguay*

J. A. Clarkson [*Uniformly convex spaces*, Trans. Amer. Math. Soc., vol. 40, 1936, pp. 396–414] introduced the following expression for the "angle" of two nonvanishing elements of any linear normed space:

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Prove that if x, y, $x+y \neq 0$, then $\alpha[x+y, x] \leqq \alpha[x, y]$.

4832. *Proposed by A. Oppenheim, University of Malaya, Singapore*

Find all integral solutions of the following Diophantine equations:

(A) $$ax^2 + ay^2 + z^2 - 2axyz - 1 = 0 \qquad (a = 1, 2, \cdots),$$

(B) $$9x^2 + 25y^2 + 49z^2 - 210xyz - 1 = 0,$$

(C) $$9x^2 + 25y^2 + 4z^2 - 60xyz - 1 = 0.$$

[The case $a = 1$ of (A) is known.]

4833. *Proposed by F. H. Northover, Carleton University, Ottawa*

In the polynomial $A(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \geqq 0$, $A(1) = 1$, and the function

$$\frac{A'(1) - xA'(x)}{(1 - x)\{1 - A(x)\}}$$

is expanded into $\sum_{i=0}^{\infty} b_i x^i$. Prove that $b_i \geqq A'(1)$ for all i.

SOLUTIONS

Real Functions

4756 [1956, 596; 1958, 534]. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $p(x_1, \cdots, x_n)$, $q(x_1, \cdots, x_n)$ be two real functions of n real variables x_i, defined and continuous in a parallelotope $R: 0 \leqq x_i \leqq a_i < \infty$. Assume that $p(x_1, \cdots) = q(x_1, \cdots) = 0$ whenever $x_1 x_2 \cdots x_n = 0$, and that $p(x_1, \cdots) > 0$, $q(x_1, \cdots) \geqq 0$ when $x_1 x_2 \cdots x_n \neq 0$. Prove there exists a real function $h(u)$ of a real variable u, defined, continuous and strictly increasing for $u \geqq 0$, $h(0) = 0$, such that throughout R, except when $x_1 x_2 \cdots x_n = 0$,

$$h\{q(x_1, \cdots)\} < p(x_1, \cdots).$$

II. *Solution by John Rainwater, University of Washington, Seattle.* The solution given in the September issue is fallacious as may be seen easily from the example: $n = 1$, $p(x) = x^2$, $q(x) = x$. $h(x)$ can be found, however, as follows.

Let $\mathbf{x} = (x_1, \cdots, x_n)$. Define the function $i(u)$ for real numbers u between 0 and the greatest value q_0 of q by $i(u) = \min \{p(\mathbf{x}) \mid q(\mathbf{x}) \geqq u\}$; for $u > q_0$, let $i(u)$ be the maximum value p_0 of p. In general i will be discontinuous, but it is increasing (hence integrable) and strictly positive for $u > 0$. Define

$$j(u) = \frac{1}{u} \int_0^u i(t)\,dt \text{ for } u > 0, \qquad j(0) = 0.$$

Evidently j is continuous and $j \leqq i$; in particular $j\{q(\mathbf{x})\} \leqq p(\mathbf{x})$ for all $\mathbf{x}$. Then let $h = j/2$.

In an arbitrary compact space the result becomes: if p and q are nonnegative functions and the set of zeros of q contains the set of zeros of p, then $p \leqq hq$ for some strictly increasing continuous h vanishing at 0.

Valid solutions also by J. Horváth and by the proposer.

Polynomial Solution of a Differential System

4783 [1958, 288]. *Proposed by Leonard Carlitz, Duke University*

Find the polynomial solution of the system

$$\begin{aligned} xf'(x) - g'(x) &= \mu f(x) \\ xg'(x) - f'(x) &= \nu g(x), \end{aligned} \tag{1}$$

where μ, ν are assigned constants.

Solution by the proposer. I. Let the degree of $f(x)$ exceed that of $g(x)$. Then the first equation in (1) implies $\deg f(x) = m = \mu$, and the second implies $\deg g(x) = m - 1$. Put

$$\begin{aligned} f(x) &= a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \cdots \qquad (a_0 \neq 0), \\ g(x) &= b_1 x^{m-1} + b_2 x^{m-2} + \cdots. \end{aligned}$$

Then the system (1) is equivalent to the system of equations

$$(m-r)a_r-(m-r+1)b_{r-1}=ma_r, \qquad (m-r)b_r-(m-r+1)a_{r-1}=\nu b_r$$

which is the same as

$$\begin{aligned} ra_r &= -(m-r+1)b_{r-1} \\ (m-r-\nu)b_r &= (m-r+1)a_{r-1} \end{aligned} \qquad (r\geqq 1). \tag{2}$$

Since $b_1\neq 0$ it follows that $a_2\neq 0$, this in turn implies $b_3\neq 0$, $a_4\neq 0$, and so on. That is, we have generally $a_{2r}\neq 0$, $b_{2r-1}\neq 0$. Moreover the second equation in (2) then yields

$$\nu \neq m-2r+1 \qquad (1\leqq r\leqq \tfrac{1}{2}m). \tag{3}$$

On the other hand it is clear that $a_1=0$, which implies $(m-2-\nu)b_2=0$, so that either $b_2=0$ or $\nu=m-2$. If $b_2=0$, then $a_3=0$ and $(m-4-\nu)b_4=0$, and so on. Thus if

$$\nu \neq m-2k \qquad (1\leqq k\leqq \tfrac{1}{2}m), \tag{4}$$

then all $a_{2r-1}=b_{2r}=0$. If, however, $\nu=m-2k$ for some k in the interval $1\leqq k\leqq \frac{1}{2}m$, we can only assert that $a_{2s-1}=b_{2s-2}=0$ $(1\leqq s\leqq k)$.

Assume first that (4) is satisfied. By means of (2) we get

$$a_{2r}=-\frac{m-2r+1}{2r}b_{2r-1}=-\frac{(m-2r+1)(m-2r+2)}{2r(m-2r+1-\nu)}a_{2r-2},$$

from which it is clear that

$$\begin{aligned} a_{2r} &= (-1)^r\frac{m(m-1)\cdots(m-2r+1)}{2^r r!(m-\nu-2r+1)(m-\nu-2r+3)\cdots(m-\nu-1)}a_0 \\ &= (-1)^r\frac{m!a_0}{2^{2r}r!(m-2r)!(\lambda+m-r)_r}, \end{aligned}$$

where $\lambda=\frac{1}{2}(1-m-\nu)$ and $(a)_r=a(a+1)\cdots(a+m-1)$, $(a)_0=1$. Now put

$$f_1(x)=\sum_{2r\leqq m}a_{2r}x^{m-2r}=\frac{a_0 m!}{2^m}\sum_{2r\leqq m}(-1)^r\frac{(2x)^{m-2r}}{r!(m-2r)!(\lambda+m-r)_r};$$

because of (3) no factor in the denominator of a_{2r} can vanish. Alternatively we may write

$$f_1(x)=\frac{a_0}{2^m}\frac{m!}{(\lambda)_m}P_m^{(\lambda)}(x), \tag{5}$$

where $P_m^{(\lambda)}(x)$ is the ultraspherical polynomial of degree m (see for example, Szegö, *Orthogonal Polynomials*, 1939, p. 84); note however that for certain λ the representation (5) breaks down.

Turning next to the b's we find that

$$b_{2r+1} = \frac{m-2r}{m-2r-1-\nu}\, a_{2r} = \frac{(-1)^r m! a_0}{2^{2r+1} r!(m-1-2r)!(\lambda+m-1-r)_{r+1}}.$$

Hence we put

$$g_1(x) = \sum_{2r<m} b_{2r+1} x^{m-2r-1} = \frac{a_0}{2^m}\,\frac{m!}{(\lambda)_m}\, P_{m-1}^{(\lambda)}(x). \tag{6}$$

Next assume that $\nu = m-2k$ for some k in the interval $1 \leqq k \leqq \frac{1}{2}m$. Then a_{2k+1} is not determined, but using (2) we get

$$a_{2k+2r+1} = \frac{(\nu-1)!}{2^r r!(\nu-1-2r)!}\,\frac{a_{2k+1}}{(2k+3)(2k+5)\cdots(2k+2r+1)}$$

$$= (-1)^r \frac{(\nu-1)! a_{2k+1}}{2^{2r} r!(\nu-1-2r)!(\lambda+\nu-1-r)_r}$$

and accordingly put

$$f_2(x) = \sum_{2r<\nu} a_{2k+2r+1} x^{\nu-2r-1} = \frac{a_{2k+1}}{2^{\nu-1}}\,\frac{(\nu-1)!}{(\lambda)_{\nu-1}}\, P_{\nu-1}^{(\lambda)}(x).$$

Similarly we get

$$b_{2k+2r} = -\frac{2k+2r+1}{\nu-2r}\, a_{2k} = (-1)^r \frac{(\nu-1)!(\lambda)_{\nu-r}}{2^{2r-1} r!(\nu-2r)!(\lambda)_{\nu-1}}\, a_{2k+1},$$

so that

$$\sum_{2r\leqq\nu} b_{2k+2r} x^{\nu-2r} = \frac{a_{2k+1}}{2^{\nu-1}}\,\frac{(\nu-1)!}{(\lambda)_{\nu-1}} + P_\nu^{(\lambda)}(x).$$

To sum up, if (4) holds, then

$$f(x) = \frac{A}{(\lambda)_m} P_m^{(\lambda)}(x), \qquad g(x) = \frac{A}{(\lambda)_m} P_{m-1}^{(\lambda)}(x) \tag{7}$$

furnish the general polynomial solution of (1); but if $\nu = m-2k$, where $1 \leqq k \leqq \frac{1}{2}m$, then the general polynomial solution may be written in the form

$$f(x) = AP_m^{(\lambda)}(x) + BP_{\nu-1}^{(\lambda)}(x), \qquad g(x) = AP_{m-1}^{(\lambda)}(x) + BP_\nu^{(\lambda)}(x), \tag{8}$$

where A and B are arbitrary constants. Note in particular that when $\nu = -m$, $\lambda = \frac{1}{2}$ and (7) reduces to $f(x) = CP_m(x)$, $g(x) = CP_{m-1}(x)$, where $P_m(x)$ is the Legendre polynomial. Also when $\nu = 0$, define $P_{-1}^{(\lambda)}(x) = 0$; however in this case (8) reduces to

$$f(x) = A(x^2 - 1)^k, \qquad g(x) = Am\int (x^2 - 1)^{k-1}dx + B.$$

II. An obvious interchange of roles disposes of the case where deg $g(x)$ $>$deg $f(x)$.

III. Let deg $f(x)$ = deg $g(x)$ = m; thus $\mu = \nu = m$. Then (1) implies

$$(x + 1)\{f'(x) - g'(x)\} = m\{f(x) - g(x)\},$$
$$(x - 1)\{f'(x) + g'(x)\} = m\{f(x) + g(x)\},$$

from which we get $f(x) - g(x) = 2A(x+1)^m$, $f(x) + g(x) = 2B(x-1)^m$. Therefore

$$f(x) = A(x + 1)^m + B(x - 1)^m, \qquad g(x) = -A(x + 1)^m + B(x - 1)^m$$

furnish the general solution in this case.

Remark. An alternative method of solving (1) is to note that (1) implies

$$(1 - x^2)f'' - (2 - \mu - \nu)xf' + \mu(1 - \nu)f = 0,$$
$$(1 - x^2)g'' - (2 - \mu - \nu)xg' + \nu(1 - \mu)g = 0.$$

However, with this method it seems more difficult to isolate the solution (8).

Also solved, more or less exhaustively, by A. P. Boblétt, J. W. Brown, P. L. Chessin, Emil Grosswald, A. R. Hyde, N. D. Kazarinoff, A. G. Konheim, A. F. Landry, W. R. Longley, Douglas Maurer, E. P. Miles, Jr., Kyu Sam Park, R. J. Pegis, Robert Spira, Max Wyman, and J. W. Young.

Property of a Sequence

4785 [1958, 289]. *Proposed by Paul Erdös, Israel Institute of Technology.*

Let $0 \leqq n_1 < n_2 < \cdots$ be a sequence of integers. Denote by $f(N)$ the number of the n's not exceeding N. Assume that $f(2N) = f(N) < c_1$. Then there is a constant c_2, depending only on c_1, such that as $x \to 1$

$$\left| \sum_{k=1}^{\infty} x^{n_k} - f\left(\frac{1}{1 - x}\right) \right| < c_2.$$

Solution by P. T. Bateman, University of Illinois. For convenience we put $x = e^{-s}$, where s is a positive real number. We shall prove that

$$(*) \qquad -2c_1 - 1 < \sum_{k=1}^{\infty} e^{-sn_k} - f\left(\frac{1}{1 - e^{-s}}\right) < c_1$$

for all positive values of x, so that c_2 can be taken as $2c_1 + 1$.

Since $e^s - 1 > s > 1 - e^{-s}$ for all positive s, we have

$$0 < \frac{1}{1 - e^{-s}} - \frac{1}{s} < 1.$$

Hence

$$0 < f\left(\frac{1}{1-e^{-s}}\right) - f\left(\frac{1}{s}\right) < 1,$$

so that we can consider $f(1/s)$ instead of $f(1/(1-e^{-s}))$. More specifically, if we put

$$\Delta(s) = \sum_{k=1}^{\infty} e^{-snk} - f\left(\frac{1}{s}\right),$$

we have

$$\Delta(s) - 1 < \sum_{k=1}^{\infty} e^{-snk} - f\left(\frac{1}{1-e^{-s}}\right) < \Delta(s). \tag{1}$$

Since

$$\sum_{k=1}^{\infty} e^{-snk} = \lim_{K\to+\infty} \sum_{k=1}^{K} e^{-snk} = \lim_{K\to+\infty} \int_0^K e^{-su}df(u)$$

$$= \lim_{K\to+\infty} \left(e^{-sK}f(K) + s\int_0^K e^{-su}f(u)du\right) = s\int_0^{\infty} e^{-su}f(u)du,$$

we obtain

$$\Delta(s) = s\int_0^{\infty} e^{-su}f(u)du - f\left(\frac{1}{s}\right) = s\int_0^{\infty} e^{-su}\left\{f(u) - f\left(\frac{1}{s}\right)\right\}du.$$

On the one hand, since f is nondecreasing, we have

$$\Delta(s) \leqq s\int_{1/s}^{\infty} e^{-su}\{f(u) - f(1/s)\}du \leqq \sum_{n=1}^{\infty} s\int_{2^{n-1}/s}^{2^n/s} e^{-su}\{f(u) - f(1/s)\}du.$$

Now by the assumption of the problem

$$f(1/s) > -c_1 + f(2/s) > -2c_1 + f(2^2/s) > \cdots > -nc_1 + f(2^n/s).$$

Hence the integrand in the nth term of the preceding sum does not exceed $e^{-su}\cdot nc_1$. Accordingly

$$\Delta(s) < \sum_{n=1}^{\infty} nc_1(e^{-2^{n-1}} - e^{-2^n}) = c_1\sum_{n=1}^{\infty} e^{-2^{n-1}} < c_1. \tag{2}$$

On the other hand

$$-\Delta(s) \leqq s\int_0^{1/s} e^{-su}\{f(1/s) - f(u)\}du$$

$$\leqq \sum_{n=1}^{\infty} s\int_{2^{-n}/s}^{2^{-n+1}/s} e^{-su}\{f(1/s) - f(u)\}du.$$

By the assumption of the problem

$$f(1/s) < c_1 + f(2^{-1}/s) < 2c_1 + f(2^{-2}/s) < \cdots < nc_1 + f(2^{-n}/s).$$

Thus the integrand in the nth term of the preceding sum is less than nc_1. Accordingly

$$-\Delta(s) < \sum_{n=1}^{\infty} s \cdot nc_1 \cdot 2^{-n}/s = 2c_1. \tag{3}$$

Combining (1), (2), and (3), we obtain (*).

Also solved by Oliver Aberth and Michael Barr, Robert Breusch, and Joe Lipman.

Editorial Note. The proposer remarks that the problem would have been more elegant if the condition $f(2N)-f(N)<c_1$ were necessary to the conclusion. The special case $n_k=k$ shows that this is not so. A similar proposition is the following: If $f(2N)/f(N)\to 1$, then as $x\to 1$,

$$\sum_{k=1}^{\infty} x^{n_k}/f\left(\frac{1}{1-x}\right) \to 1.$$

Generalization of Bergstrom's Inequality

4786 [1958, 289]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let A, B be two positive definite Hermitian matrices of order n, and let $C=A+B$. For any positive integer $p<n$, let A_p denote the principal submatrix of A formed by the first p rows and columns, and let B_p, C_p have similar meaning. Prove

$$\left(\frac{\det C}{\det C_p}\right)^{1/(n-p)} \geqq \left(\frac{\det A}{\det A_p}\right)^{1/(n-p)} + \left(\frac{\det B}{\det B_p}\right)^{1/(n-p)}. \tag{1}$$

[The case $p=n-1$ is known and due to H. Bergström (See R. Bellman, this MONTHLY, vol. 62, 1955, pp. 172–173). Also the inequality of Minkowski (Hardy, Littlewood and Pólya, *Inequalities*, 1934, p. 35) may be regarded as the case $p=0$.]

Solution by the proposer. It is possible to derive (1) from Bergström's inequality, *i.e.*, to derive the general case from the special case $p=n-1$. The following proof, however, avoids the need of first establishing the special case.

The proof is based on the following minimum property: *Let H be a positive definite Hermitian linear transformation in the unitary n-space U^n. Let $e_1, \cdots, e_n$ be n orthonormal vectors in U^n and let p be a positive integer $<n$. Then*

$$\frac{\det H}{\det ((He_j, e_i))_{1\leq i,j\leq p}} = \min \det ((Hx_i, x_j))_{p+1\leq i,j\leq n},$$

where the minimum is taken over all systems of $n-p$ vectors $x_{p+1}, x_{p+2}, \cdots, x_n$ in U^n subject to the biorthonormal condition

$$(x_i, e_j) = \delta_{ij} \qquad (p+1 \leqq i, j \leqq n). \tag{2}$$

This result was proved in K. Fan, *Proc. Cambridge Philos. Soc.*, vol. 51, 1955, pp. 414–421.

To prove (1), let $e_1, \cdots, e_n$ denote the unit vectors of our coordinate system in U^n. Then

$$A_p = ((Ae_j, e_i))_{1 \leqq i,j \leqq p}, \qquad B_p = ((Be_j, e_i))_{1 \leqq i,j \leqq p},$$
$$C_p = ((Ce_j, e_i))_{1 \leqq i,j \leqq p}.$$

According to the minimum property stated above, there exist $n-p$ vectors $x_{p+1}, \cdots, x_n$ satisfying (2) and

$$\frac{\det C}{\det C_p} = \det ((Cx_i, x_j))_{p+1 \leqq i,j \leqq n}. \tag{3}$$

Let D, E, F denote the matrices of order $n-p$:

$$D = ((Ax_i, x_j))_{p+1 \leqq i,j \leqq n}, \qquad E = ((Bx_i, x_j))_{p+1 \leqq i,j \leqq n},$$
$$F = ((Cx_i, x_j))_{p+1 \leqq i,j \leqq n}.$$

Again by the minimum property, we have

$$\det D \geqq \frac{\det A}{\det A_p}, \qquad \det E \geqq \frac{\det B}{\det B_p}. \tag{4}$$

The matrices D, E, F are clearly Hermitian and positive definite. Now, applying Minkowski's inequality to $F=D+E$, we get

$$(\det F)^{1/(n-p)} \geqq (\det D)^{1/(n-p)} + (\det E)^{1/(n-p)}. \tag{5}$$

From (3), (5), and (4), (1) now follows immediately.

An Incorrect Proposal

4787 [1958, 289]. *Proposed by M. S. Klamkin, AVCO Research Division, Wilmington, Mass.*

Express as a single definite integral

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(m+n+r)!m^n n^r}{m!n!r!(r+1)^{m+n+r+1}}.$$

Editorial Note. As several readers were quick to point out, the stated series is divergent. Attempts to rework the original idea into a well-posed problem have not been satisfactory. The editor and the proposer apologize for a careless oversight.

Omitted Values of Entire Functions

4788 [1958, 370]. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Let $f(z)$ be an entire function and consider the two statements: (1) If f, f', f'' never vanish, then $f(z) = e^{az+b}$. (2) If f, f', f'' each miss a value, then $f(z) = a + be^{cz}$. (1) is a known theorem. (2) is apparently stronger than (1). Show, however, that (2) is an elementary consequence of (1).

I. *Solution by the proposer.* We assume (1) and proceed to prove (2). Suppose then f is given where f misses the value a, f' misses ξ, f'' misses η. We distinguish three cases:

Case I. $\xi \neq 0$. Form the function

$$F = e^{\xi \int 1/(f-a)}.$$

We find that

$$F' = \frac{\xi}{f-a} e^{\xi \int 1/(f-a)}, \qquad F'' = \frac{\xi(\xi - f')}{(f-a)^2} e^{\xi \int 1/(f-a)},$$

and that no one of F, F', F'' is ever zero. (1) applied to F tells us that $F = e^{\alpha z+\beta}$ and hence f must be a constant. (2) is trivially true in this case.

Case II. $\eta \neq 0$. A similar argument with

$$F = e^{\eta \int 1/(f'-\xi)}$$

proves that f' is a constant so that f is linear. Since f misses a value, however, it must be constant and again (2) follows.

Case III. $\xi = 0$, $\eta = 0$. (2) follows upon applying (1) to $f - a$.

II. *Solution by D. Gaier, Technische Hochschule, Stuttgart.* Saxer proved (1923) that if $f(z)$ is an entire function and $f(z)$ and $f'(z)$ miss the values A and B, respectively, then $B = 0$. Hence, if $f(z)$, $f'(z)$, $f''(z)$ miss the values A, B, C, respectively, and $f(z)$ is not constant, then $B = C = 0$ and by proposition (1), $f(z) - A$ is of the form e^{az+b}.

It is possible to improve the stated result. If $f(z)$ is an entire function and $f(z)$ and $f''(z)$ miss the values A and C, then $f(z) = A + e^{az+b}$. In fact, Saxer's theorem has been generalized by Bureau (1931) who replaced $f'(z) \neq B$ by $f^{(k)}(z) \neq B$, $(k \geq 1)$ with the same conclusion, $B = 0$. Hence $f(z) - A$ and $f''(z)$ each miss the value zero, so that by a recent result of Hayman (1958), $f(z) - A$ is of the form e^{az+b}.

Also solved by James Clunie.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The regulations recently adopted by the Office of Education on the administration of Title III of the National Defense Education Act of 1958 seem to spell out quite clearly that *these funds may be used to purchase books in mathematics for school libraries.* Apparently many administrators are unaware that "the acquisition of supplementary reading materials to strengthen the mathematical portion of school libraries" is an eligible and important project under the Act. If mathematics teachers are to take advantage of this vital opportunity, they must act vigorously and swiftly at both the local and the state levels. Please do call this provision to the attention of *your* administrative officers. The high school book list prepared by the National High School and Junior College Math Club is still available to persons sending a stamped self-addressed envelope (number 10 size) to HIGH SCHOOL BOOK LIST, c/o Richard V. Andree, The University of Oklahoma, Norman, Oklahoma.

Convexity. By H. G. Eggleston. Tracts in Mathematics and Mathematical Physics No. 47, Cambridge, 1958. viii+136 pp. $4.00.

In this tract the author presents an introduction to convex sets in finite dimensional euclidean space. Since there had been nothing previously available in book form in English on convexity, this book will be very useful. All the topics chosen, except those in chapters 6 and 7, are virtually necessary to the subject. Included are Helly's theorem, mixed volumes, Blaschke's compactness theorem, the Brun-Minkowski inequality, supporting functions, gauge functions and so on. Chapter 6 is devoted to special problems and chapter seven to sets of constant width.

The author has performed a valuable service to the mathematical community in making the early results on convex sets available. However, in doing this, one might object that some items of value have been developed since 1930 and might have been included. For example, the semispaces the reviewer introduced some years ago are more important to the basic theory of convex sets than are the halfspaces. None the less, an introduction to convex sets is now made accessible to undergraduate students by virtue of this publication.

PRESTON C. HAMMER
University of Wisconsin

Modern Business Statistics. By John E. Freund and Frank J. Williams. Prentice-Hall, Englewood Cliffs, N. J., 1958. xv+539 pp. $10.00.

One school of thought insists that basic statistics is basic statistics and that there is little to be gained, usually much to be lost, by offering introductory

courses in educational statistics, agricultural statistics, *etc.* The reviewer accepts this philosophy and accordingly sees little merit in this volume. He objects to the word "modern" in the title, for the material is traditional, and we even find those twin Ulyssean perils of ancient statistics, platykurtic and leptokurtic, introduced on page 107.

For the unenlightened (those who take the opposite point of view), it might be admitted that this book is better than most of its competitors in the "applied" class. Indeed, chapters VI–XIV constitute a brief but entirely satisfactory introduction to basic statistical concepts, and it is gratifying to find mention of such ideas as conditional probability. The most serious omission here is a chart for testing the significance of the sample correlation coefficient.

Unfortunately, the remaining half of the book is not of the same quality. Thus 120 pages are devoted to a routine discussion of index numbers and time series, and in the first 42 pages the student will essentially only learn the meanings of $\sum$, histogram, and class frequency. There are also some minor slips such as the phrase "absolute certain" on page 112 and the implication that the customary definition of s is unbiased on page 86.

The authors do not offer the teacher overly much choice in problem material. Generally speaking, the problems are not very stimulating although some of the examples have interesting business pedigrees.

K. A. Bush
University of Idaho

Preparatory Mathematics, Vols. I and II. By Georges, Sunko, Eulenberg, Piety. Edwards, Ann Arbor, 1957 and 1958. 208 and 209 pp. $4.00 and $4.50.

These volumes are "designed to meet the mathematical needs" of college freshmen who either "have had neither algebra nor geometry in high school" or "who score low on the placement test administered by the Department of Mathematics."

The subjects listed in the Contents range from elementary arithmetic through numerical trigonometry.

The first half of each volume consists primarily of many declarative sentences. Practically no attempt is made to have the student understand what he is doing. The entire discussion of the problem of finding the area of a circle consists of the following: "$A=\pi r^2$, where $\pi=3\frac{1}{7}$, 3.14, 3.1416, *etc.*" At another point we find the statement "a power of 10 which is a multiple of 3 is a perfect cube."

The second half of each volume consists of tear sheets on which the student, following the directions given in the corresponding section of the first half, inserts numbers in the proper boxes.

The student who uses these volumes will have much practice in writing neatly. This reviewer doubts that he will learn any mathematics.

Lloyd L. Lowenstein
Arizona State University

Selections from Modern Abstract Algebra. By Richard V. Andree. Holt, New York, 1958. xii+212 pp. $6.50.

This book differs from recent texts which are introductions to modern algebra in several respects. It is intended for an early place in the mathematics curriculum, namely at the sophomore level for mathematics majors, and possibly somewhat later for science and engineering students. It makes no pretense at completeness for the topics included and does not emphasize any one facet of modern algebra, but rather offers short selections from various parts of the subject. It does not presuppose much mathematical maturity and, in fact, one of the basic purposes of the book is to develop this maturity.

In Chapter I, the postulates for an integral domain are given as an abstraction of the arithmetical properties of the integers. Undefined terms, postulates, definitions, and the nature of proof are discussed briefly. Examples of finite number systems are given, and some elementary number theory is included.

Chapter II is devoted to equivalence relations and integral congruences, and Chapter III discusses binary Boolean arithmetic with its application to circuit network theory.

Chapter IV introduces the concept of a group and numerous examples are given. The elementary properties of a group are derived up to the notions of invariant subgroup and quotient group.

Chapter V contains the basic algebra of matrices, and Chapter VI applies this material to the solution of systems of linear equations. Chapter VII gives a fairly standard account of the properties of determinants.

Chapter VIII gives the ring and field postulates. Ideals and residue class rings are defined.

The final Chapter IX makes a selection from further topics in matric theory, discussing briefly the characteristic equation, characteristic roots and vectors, the minimum function of a matrix, infinite series with matric elements, and derivatives and integrals of matrices with elements which are functions of one variable.

The misprints which this reviewer noted are not of serious nature and should be easily spotted by a careful reader. There are adequate problem sets after each section, and an interesting feature of the book is that many problems consist of suggested reading from the AMERICAN MATHEMATICAL MONTHLY.

R. A. BEAUMONT
University of Washington

Editorial Note. Since Professor Andree is the editor of this section, the editor-in-chief has assumed responsibility for the editing of this review. R.D.J.

Intermediate Analysis. By John M. H. Olmsted. Appleton-Century-Crofts, New York, 1956. xii+305 pp. $6.00.

The title of this book does not suggest its content; the area treated is better described by its subtitle, ***An Introduction to the Theory of Functions of One Real***

Variable. The author's stated purpose is "to present the basic ideas and techniques of analysis, for functions of a single real variable, in such a way that students who have studied calculus can proceed at whatever pace and intensity are considered suitable," and he has quite definitely accomplished his aim.

The author presents in a highly readable and extremely careful fashion *all* of the purely mathematical topics concerning functions of a single real variable usually found in an American calculus text, as well as material on the most elementary point-set topology, uniform continuity, functions of bounded variation, Cauchy criteria for the convergence of sequences and functions, some special convergence tests for series, the doppelreihensatz, the Cauchy product of series, and uniform convergence. Many topics not treated in the main body of the book are covered in a large collection of exceedingly good, graduated exercises: The relation between Riemann integrability and continuity almost everywhere and some of its consequences (dominated and bounded convergence theorems for Riemann integrals), uniform approximation of continuous functions by polynomials, the partial fractions decomposition theorem, and the Moore-Osgood theorem on interchange of limits, to mention but a very few.

Some flexibility in the use of the book has been considered by the author. He has organized the material so that certain starred portions may be omitted without affecting the continuity of the book as a whole, but the reviewer feels that some caution should be exercised in this direction. To cite but one case in point, omission of the axiom of completeness for the real number system seems to defeat the very purpose of the book as a deeper study of the calculus, since all of the latter is based on that axiom!

In the reviewer's opinion, this book would be a superb text for a one-quarter, five-credit (one-semester, three-credit) course at the junior or senior level to complement a course in differential equations and thereby afford a good year-sequence in elementary analysis. Further, every math major (including high school teachers and teacher majors), whose advanced mathematical training leans away from the field of analysis, would gain immeasurably by the presence of this book in his personal library.

The reviewer could find only one feature about which to complain: The author's definition of a function is the time-honored one of a relation.

ARTHUR E. LIVINGSTON
University of Washington

Mathematical Foundations of Information Theory. By A. I. Khinchin, translated by R. A. Silverman and M. D. Friedman. Dover, New York, 1957. vii+120 pp. $1.35.

The subject of information theory is very new and has been developed primarily by mathematicians in this country. The basic ideas were set forth by C. Shannon in 1948, and extended in fundamental papers by B. McMillian and A. Feinstein.

There have been a number of expository treatments of information theory, but these have usually been based on ill-defined concepts and accompanied by rather exaggerated claims about the universal applicability of the subject.

The book under consideration is a translation of two papers written by the Russian mathematician, A. I. Khinchin, for the expository journal *Uspekhi*. These papers present the mathematical foundations of information theory. While completely rigorous, the flavor of the engineering applications which led to the theory runs throughout and very much helps the intuition. Khinchin has here reformulated basic concepts and presents for the first time rigorous proofs of certain fundamental theorems in the subject.

The first paper discusses the concept of entropy and gives one major application to coding. The only stochastic processes used are Markov chains. This paper would serve as a valuable supplement to an introductory probability course.

The second and longer paper uses more advanced topics from probability theory—for example, stationary processes and martingales. However, the treatment is quite complete and the nonspecialist would not suffer, thanks to Khinchin's amazing expository ability.

It is a tribute to Shannon's theory that a rigorous treatment only enhances the elegance of the basic theorems.

J. Laurie Snell
Dartmouth College

An Introduction to Euclidean Geometry. By J. C. Eaves and A. J. Robinson. Addison-Wesley, Reading, Massachusetts, 1957. xii+327 pp. $4.25.

This interesting workbook-style text (with soft cover and 8½ by 11 inch pages) is designed primarily to meet the needs of students who enter college with inadequate high school preparation in geometry. In the thirty-seven lessons into which the book is divided, the authors have presented those concepts of plane and solid geometry which will give adequate background to the student who intends to major in science or engineering. The material can be covered in a semester three- or four-hour course.

The book is well written and is very attractive, with its many excellent drawings and reproductions of photographs. Students are carefully led to the point where they can fill in many of the details of the proofs in the text portions of the lessons. At the end of each lesson is a "summary for easy reference," which the student is encouraged to amplify and use for review purposes. Sets of exercises are on perforated sheets which can be torn out and handed in.

Plane geometry is covered in the first twenty-five lessons. In these lessons are found twenty-two postulates, followed by ten constructions and forty-eight theorems. Other theorems are included in the 150 exercises. The pace seems a little faster in the twelve lessons on solid geometry. These lessons introduce four postulates, sixty-eight theorems, and eighty-five exercises. Many of the

theorems of three-space are stated without proof, and it is not intended that the student prove all of them.

This book is highly recommended to those who find it necessary to offer an elementary geometry course for college freshmen.

VIOLET HACHMEISTER LARNEY
State University of New York
College for Teachers, Albany

Linear Algebras. Publication 502, National Academy of Sciences—National Research Council. Washington, D. C., 1957. v+59 pp. $1.50.

This is a report of a Conference on Linear Algebras held on Long Island in June 1957. A preface by A. A. Albert briefly describes the Conference and mentions several papers given at the Conference but published elsewhere. A list of the 23 participants is given at the end of the booklet. The following seven papers are published in the booklet: Irving Kaplansky, *Problems in the theory of rings;* R. Brauer, *Some remarks on associative rings and algebras;* N. Jacobson, *Jordan algebras;* Erwin Kleinfeld, *On alternative and right alternative rings;* George B. Seligman, *A survey of Lie algebras of characteristic p;* Reinhold Baer, *Meta ideals;* David Buchsbaum, *A Survey of homological algebra.*

As the titles indicate, several of the papers survey the literature and have fairly extensive bibliographies. On the other hand, Kaplansky states twelve outstanding problems; Brauer generalizes the Artin-Wedderburn structure theory; Kleinfeld disposes of simple alternative rings; and Baer introduces a topic new to ring theory. Since detailed individual reviews of the papers presumably will appear elsewhere, we shall not peer more closely. Let us say merely that the booklet seems well worth owning.

R. H. BRUCK
University of Wisconsin

Vector Spaces and Matrices. By Robert M. Thrall and Leonard Tornheim. Wiley, New York, 1957. xii+318 pp. $6.75.

The past few decades have witnessed a growing interest in vector spaces and matrices, with the juxtaposition of these topics indicating that matrices have come to be recognized for what they are. Even more noteworthy, perhaps, is the fact that these topics have descended from the level of esoterica to that area which is traditionally the citadel of reaction and conservatism: the undergraduate mathematics curriculum. The book under review is the latest in an ever-lengthening series of texts treating vectors and matrices in the modern flavor and is, apparently, intended for the undergraduate market.

There are nice things to be said about this compendium. Noteworthy is the authors' announced intention to proceed on two levels—one concrete and the other abstract—the one via matrices and the other via linear transformations. This program is maintained, more or less, for the first six chapters, and serves to

clarify the theory. An item deserving more attention still is the beautiful development of chapters 8 through 10: this is not easy reading, but the result is the complete canonical form under similarity in the general case, and the reader can feel satisfied that he has seen the whole exposition before his very eyes.

The reviewer's main criticism is that it is difficult to decide for whom the book was written, since the level of exposition varies markedly from page to page. For instance, on page 25 the following lemma is proved: Let W be a subspace of a finite-dimensional vector space V over F. Then $W=V$ if and only if dim $W=$dim V. The authors, without further discussion of cardinality, remark that the lemma is false for infinite-dimensional spaces, "as can be seen by taking $F\equiv$rationals, $W\equiv$reals and $V\equiv$complexes." As the beginning student recovers from this remark, he gets to page 37, where the summation symbol is patiently explained, and to page 38, where three illustrations of the meaning of the symbol are written out. But probably the most startling page in the text is page 61. For hereon there are defined the concepts of group, abelian group, ring, commutative ring with identity, algebra over a field, and total matric algebra of degree v.

Other such examples of the hill-and-valley school of writing could be listed, but the point has undoubtedly been made. In the reviewer's opinion the book is worth the price for chapters 8–10, for use as one of many supplementary texts, by beginning graduate students. This class of reader might also be interested in the last chapter, Linear Programming and Game Theory, although it is only tenuously related to the rest of the book and seems isolated and out of place.

As an undergraduate text the book is uneven and certainly not appealing to this reviewer. However, intrepid souls will certainly want to try it and, for these, the reviewer can but paraphrase the old aphorism: Let the prospective teacher beware!

R. L. San Soucie
Sylvania Electronic Systems
Buffalo, New York

Applied Differential Equations. By Murray R. Spiegel. Prentice-Hall, Englewood Cliffs, N. J., 1958. xv+381 pp. $6.75.

This book is an excellent one for students who are studying differential equations for the first time and who are primarily interested in applications to engineering, physics and chemistry.

Well-chosen examples are used to motivate each basic idea as it is introduced. Then the theory underlying the idea is developed, with due regard to rigor. Finally the idea is exploited through a variety of applications; the treatment of the applications emphasizes the formulation of the problem, the solving, and the interpretation of the solution. The A exercises are straightforward, the B exercises are more difficult, while the C exercises are intended to challenge the student and at the same time to open up a number of new possibilities to him.

The book has been well conceived and should effectively achieve its purposes. Besides using it with a regular class, one could recommend it to anyone who wants to study, or restudy, elementary differential equations and their applications.

About a third of the book is devoted to first-order and simple higher-order equations and their applications. Then follows a chapter on linear differential equations with constant coefficients, and another on their applications. There is then one chapter each on simultaneous differential equations and their applications, series solutions, numerical methods, how partial differential equations arise, and Fourier-series solutions of boundary value problems.

The chapter on numerical methods is not quite as well presented as the others. My only other criticism is that the book contains a few minor inaccuracies and a number of statements which need to be either modified or clarified.

T. E. HULL
The University of British Columbia

BRIEF MENTION

Functions of Real and Complex Variables. By William F. Osgood. Chelsea, New York, 1958. xii+407 pp. and viii+262 pp. $4.95.

It is welcome news for many to find Osgood's 1935 and 1936 books bound together in one volume at this low price. In spite of the many books which have been written on these two subjects during the last two decades, Osgood's works still have much to recommend them to the student emerging from a course in advanced calculus to take his first real look at analysis.

Elementary Number Theory. By Edmund Landau. Chelsea, New York, 1958. 256 pp. $4.95.

This is an English translation of Landau's *Elementare Zahlentheorie* by J. E. Goodman, which has been supplemented by exercises contributed by Professors Bateman and Kohlbecker. The material is, of course, in no sense modern, but still is worthy of a student's time. Part I contains the foundations of elementary number theory including quadratic residues and Pell's equation. Part II deals with the theorems of Brun and Dirichlet. Part III discusses decomposition into squares while the fourth portion is devoted to binary quadratic forms and their class numbers.

Logic Machines and Diagrams. By Martin Gardner. McGraw-Hill, New York, 1957. ix+157 pp. $5.00.

The skilled pen of Martin Gardner has now turned to a discussion of logic, mainly using geometric forms. This interesting blend of history, mathematics, and just plain fun is combined with an occasional rather sharp barb. (See for example page 59, number 5.) Certainly the book deserves a place on the shelf of anyone interested in elementary logic machines.

The Teaching of Modern School Mathematics. By E. J. James. Oxford University Press, New York, 1958. viii+275 pp. $3.40.

A pedagogical type of book, dealing with the instruction in the English modern secondary school. The majority of the work discussed would have sounded quite familiar to a seventh or eighth grade mathematics teacher in the American school system of 15 or 20 years ago.

Science in a Tavern. By Charles S. Slichter. University of Wisconsin Press, 1958. ix+206 pp. $1.00.

A reprint of Professor Slichter's delightful essays on science-in-the-making which first appeared some twenty years ago. This inexpensive little volume belongs on the bookshelf of anyone who habitually thinks of a scientist in terms of a large well-equipped modern laboratory.

Dilogarithms and Associated Functions. By L. Lewin. Macdonald, London, 1958. xvi+353 pp. 65s.

A book which requires, essentially, only elementary calculus as a prerequisite. This is probably the first extensive publication on dilogarithms since they were introduced almost a hundred and fifty years ago.

Linear Programming and Associated Techniques. By Vera Riley and Saul I. Gass. Johns Hopkins, Baltimore, 1958. x+613 pp. $6.00.

This is not a text but a comprehensive bibliography on linear programming, nonlinear programming and dynamic programming. Each publication contains its complete title, source and a brief descriptive paragraph. This volume contains over twice as many items as that of its 1954 predecessor and should be welcomed indeed by mathematicians interested in linear programming.

Hammond's Ambassador World Atlas. C. S. Hammond and Co., Maplewood, New Jersey, 1954. 416 pp. $12.50.

An excellent collection of large maps of both physical and human geography. A mathematician, naturally, wishes that somewhere among the more than 400 pages it had been possible to include half a page describing the various projections used throughout the book.

Sound Pulses. By F. G. Friedlander. Cambridge University Press, New York, 1958. xi+202 pp. $7.50.

A text on aperiodic disturbances with clearly-defined fronts, based on linear partial differential equations mainly of hyperbolic type.

Engineering Systems Analysis. By Robert L. Sutherland. Addison-Wesley, Reading, Mass. 1958. xii+223. $7.50.

An undergraduate text of interest primarily because it points up the simi-

larity between topics in different fields of engineering. A calculus background is assumed.

The Principles of Science (A Treatise on Logic and Scientific Method). By W. Stanley Jevons. Dover, New York, 1958. liii+786 pp. $2.98.

Many of Jevons' ideas on logic are still of more than historical interest even though this volume was first published in 1873.

A Mathematics Dictionary. By Bristow (Oklahoma) High School chapter of Mu Alpha Theta. 46 (8½ by 11) pp. $1.00 postpaid.

Our sincere congratulations to this group of high school students and their teacher, Neva Gurley, who have constructed this carefully-compiled dictionary of more than 600 mathematical terms likely to be encountered in grades 9–12. From *abacus* and *Abelian group* to *zero* and *zone* they have done an excellent job of giving understandable definitions that do not seem stilted. Their definitions are more accurate than those in the commercially published mathematical dictionary recently reviewed in this journal. It is a pleasure to receive publications of this type.

How to Study, How to Solve. Second Edition. By H. M. Dadourian. Addison-Wesley, Reading, Mass., 1958. iv+43 pp. $0.50.

A slight revision of Dadourian's 1947 book of the same title. A welcome reference for any college or high school student who is having difficulty in mathematics.

Tables of Modified Quotients of Bessel Functions of the First Kind for Real and Imaginary Arguments. By Morio Onoe. Columbia University Press, 1958. 338 pp. $12.50.

Computations performed chiefly on the I.B.M. 607 with auxiliary computations on the I.B.M. 602 and 650 were tabulated on the I.B.M. 407 and processed by photo-offset for this publication. From eight to ten significant digits have been kept in the tables. An introductory section includes a collection of formulae and the general methods used in the preparation of the tables.

Handbook of Automation, Computation and Control, Volume I: *Control Fundamentals.* Eugene M. Grabbe, Simon Ramo and Dean E. Wooldridge, Editors. Wiley, New York, 1958. xx+972 pp. $17.00.

This amazing handbook has been prepared by a staff of specialists. The section on general mathematics, for example, was edited by R. M. Thrall and W. Kaplan. However, individual chapters have been written by R. M. Thrall, R. C. Lyndon, G. E. Hay, W. Kaplan, E. H. Rothe, A. H. Copeland, Sr., and A. B. Clarke, and include the following topics: Sets and Relations, Algebraic Equations, Matrix Theory, Finite Difference Equations, Differential Equations, Integral Equations, Complex Variables, Operational Mathematics, Laplace Transforms, Conformal Mapping, Boolean Algebra, Probability, and Statistics. The second portion is devoted to Numerical Analysis, with such well-known authors as Bernard Dimsdale, M. Mannos, J. M. Cameron, R. F. Clippinger, J. B. Diaz, Bernard Friedman, Eugene Isaacson, and Robert Richtmeyer, and is edited by Richard F. Clippinger and Joseph H. Levin. The remainder of the book is devoted to what appear to be equally competent presentations of Operations Research, Information Theory and Transmission and Feedback Control, including an excellent discussion of noise problems. All in all, this handbook certainly belongs on the shelf of every computation center and in the library of any mathematician seriously interested in computation.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

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Please do not throw away this issue. If you do not regularly file or pass your copies on after reading them, why not participate in the Magazines for Friendship program? This nonprofit organization will provide you with selected names of foreign scholars, teachers, universities, and libraries eager to receive learned U. S. publications, even old ones. For complete details, please send a stamped, self-addressed envelope to Magazines for Friendship, Occidental College, Los Angeles 41, California.

PERSONAL ITEMS

Professor M. F. Smiley, State University of Iowa, represented the Association at the Inauguration of President J. E. McCabe of Coe College, Cedar Rapids, Iowa, on December 5, 1958.

Mr. G. L. Alexanderson, Stanford University, has been appointed Instructor at the University of Santa Clara.

Assistant Professor R. E. Anderson, Northern Illinois University, has been appointed Assistant Professor at San Jose State College.

Mr. W. L. Anderson, West Virginia University, has accepted a position with the Department of Interior, Geological Survey, Washington, D. C.

Mr. M. T. Battles, Jr., University of Notre Dame, has accepted a position as Research Engineer B with the Boeing Airplane Company, Renton, Washington.

Professor R. C. Blackwell, Furman University, has been promoted to Chairman of the Mathematics Department.

Mr. D. J. Boedicker, U. S. Army, has accepted the position of Mathematical Programmer at the I.B.M. Corporation, Owego, New York.

Dr. H. F. Bright, General Electric Company, Plainville, Connecticut, has been appointed Professor of Statistics at George Washington University.

Mr. G. C. Bush, Massachusetts Institute of Technology, has been appointed Instructor at Queen's University, Kingston, Ontario.

Dr. T. C. Chen, I.B.M. Research Laboratories, Poughkeepsie, New York, has been appointed Staff Physicist in the Department of Mathematics, I.B.M. Corporation Research Center, Yorktown Heights, New York.

Mr. W. J. Cody, Jr., University of Oklahoma, has been appointed Instructor at Northwestern University.

Mr. R. W. Cottle, Harvard University, has been appointed Teacher at Middlesex School, Concord, Massachusetts.

Mr. W. J. Fleming, Jr., University of Buffalo, has been appointed Group Leader in Aerodynamics at Lockheed Missile Systems Division, Sunnyvale, California.

Dr. Duncan Harkin, U. S. Naval Observatory, Washington, D. C., has been appointed Professor at the U. S. Naval Academy.

Dr. T. P. Higgins, Dalmo-Victor, Belmont, California, has accepted a position with the Boeing Scientific Research Laboratory.

Mr. R. H. Hopkins, Mississippi State College, has been appointed Assistant Professor at Florida Southern College.

Mr. A. R. Hyde, Emhart Manufacturing Company, Hartford, Connecticut, has been appointed Master at Kingswood School, West Hartford, Connecticut.

Mr. J. H. Jordan, University of Oregon, has been appointed part-time Instructor at the University of Colorado.

Dr. Maurice Kennedy, University College, Dublin, Ireland, has been appointed Instructor for 1958–59 at Stanford University.

Dr. K. S. Kretschmer, Carnegie Institute of Technology, has accepted the position of Mathematical Economist with Imperial Oil Ltd., Toronto, Ontario.

Professor John Kronsbein, Evansville College, has been appointed Professor of Engineering at the University of Florida.

Mr. Robert Luippold, New York State College for Teachers at Albany, has been promoted to Associate Professor.

Assistant Professor Marian A. Moore, on leave from Southern Illinois University, has been appointed Visiting Professor at Hiram College.

Associate Professor M. E. Munroe, University of Illinois, has been appointed Professor and Chairman of the Mathematics Department at the University of New Hampshire.

Assistant Professor F. G. O'Brien, Fordham University, has been appointed to the staff of the Division of Scientific Personnel and Education, National Science Foundation.

Dr. Daniel Orloff, Cornell Aeronautical Laboratory, Buffalo, New York, has accepted a position as Director of the Information Processing and Computer Division of the Dasol Corporation of New York.

Dr. R. R. Phelps, University of Washington, has been appointed a Member of the Institute for Advanced Study.

Mr. F. A. Phillips, Aberdeen Proving Gound, Aberdeen, Maryland, has been appointed Senior Engineer at the Martin Company, Orlando, Florida.

Dr. D. H. Potts, U. S. Navy Electronics Laboratory, San Diego, California, has been appointed Associate Professor at Long Beach State College.

Associate Professor Irving Reiner, University of Illinois, has been promoted to Professor. He is directing a research project supported by a grant from the Office of Naval Research.

Mr. E. W. Robinson, Oklahoma Agricultural and Mechanical College, has been appointed Instructor at the University of Wichita.

Mr. W. J. Roesler, University of Minnesota, has been appointed Staff Member of Group 67, Massachusetts Institute of Technology, Lincoln Laboratories.

Associate Professor C. E. Rusch, Wisconsin State College, Eau Claire, has been appointed Professor at State University of New York, Teachers College at Oneonta.

Mr. J. D. Rutledge, Cornell University, has been appointed Staff Mathematician at the I.B.M. Corporation, Yorktown Heights, New York.

Assistant Professor L. A. Schmittroth, Montana State University, has been appointed Supervisor, Applied Mathematics Section, Phillips Petroleum Company, Idaho Falls, Idaho.

Dr. M. A. Shader, I.B.M. University Program, New York City, has been appointed Manager, Engineering and Sciences, Industry Marketing, I.B.M. Corporation, White Plains, New York.

Mr. J. F. Sheppard, University of Oklahoma, has accepted the position of Programmer with the Boeing Airplane Company, Wichita, Kansas.

Mr. R. L. Shobe, Alabama Polytechnic Institute, has accepted the position of Aerophysics Engineer with Convair, Fort Worth, Texas.

Associate Professor Jack Silber, Roosevelt University, has been promoted to Professor of Mathematics and Acting Assistant to the Dean of Faculties.

Mr. D. L. Silverman, University of California, Los Angeles, has accepted the position of Mathematician with the National Security Agency, Fort Meade, Maryland.

Mr. R. K. Smith, Boeing Airplane Company, Seattle, Washington, has been ap-

pointed Member of the Technical Staff at Space Technology Laboratories, El Segundo, California.

Mr. L. R. Stewart, Jr., I.B.M. Corporation, Poughkeepsie, New York, has been appointed Senior Applied Science Representative for the I.B.M. Corporation, Washington, D. C.

Mr. R. W. Stewart, University of Arizona, has accepted the position of Electrical Engineer at Nortronics, Anaheim, California.

Dr. D. L. Thomsen, Jr., I.B.M. Corporation, Yorktown Heights, New York, has been appointed Data Processing Manager with the I.B.M. Corporation, Watson Scientific Computing Laboratory at Columbia University.

Dr. D. B. J. Tomiuk, Catholic University of America, has been appointed Assistant Professor at Fordham University.

Mr. J. R. VanAndel, Burroughs Corporation, Paoli, Pennsylvania, has accepted the position of Senior Engineer with Bendix Research Laboratories, Detroit, Michigan.

Assistant Professor S. I. Vrooman, University of Pittsburgh, has been appointed Associate Professor at the Pratt Institute.

Associate Professor J. H. Wahab, Georgia Institute of Technology, has been appointed Professor at Louisiana State University, New Orleans.

Dr. F. J. Weyl, Naval Analysis Group, has been appointed Research Director of the Office of Naval Research, Washington, D. C.

Dr. R. E. Wild, Douglas Aircraft Corporation, Santa Monica, California, has been appointed Assistant Professor at the University of Arizona.

Dr. J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, New York, has been promoted to Assistant Manager.

Mr. G. K. Williams, University of Kentucky, has been appointed Assistant Professor at Madison College.

Mr. H. C. Arnold, Federal Enamel & Stamping Company, McKees Rocks, Pennsylvania, died on September 27, 1958. He was a member of the Association for sixteen years.

Associate Professor J. P. Brewster, Clemson Agricultural College, died on October 14, 1958. He was a member of the Association for ten years.

Mr. W. J. Ettinger, Lombard, Illinois, died on January 21, 1958. He was a member of the Association for twenty-eight years.

Professor K. P. Williams, Indiana University, died on September 25, 1958. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1958–59

Tom Apostol

New Mexico Institute of Mining and Technology	Socorro, N. M.	Mar. 16–17
New Mexico College of Agriculture and Mechanical Arts	State College, N. M.	Mar. 19–20
Univ. of Arizona	Tucson, Ariz.	Mar. 23–24
Arizona State University	Tempe, Ariz.	Mar. 26–27

San Diego State College	San Diego, Calif.	Mar. 30–31
Fresno State College	Fresno, Calif.	Apr. 2–3
San Jose State College	San Jose, Calif.	Apr. 6–7
San Francisco State College	San Francisco, Calif.	Apr. 8–9
Chico State College	Chico, Calif.	Apr. 10
University of Oregon	Eugene, Ore.	Apr. 13
Willamette University	Salem, Ore.	Apr. 14–15
Linfield College	McMinnville, Ore.	Apr. 16–17
Seattle University	Seattle, Wash.	Apr. 20–24
University of Washington	Seattle, Wash.	Apr. 20–24
Western Washington College of Education	Bellingham, Wash.	Apr. 20–24
University of Idaho	Moscow, Idaho	Apr. 27–29
State College of Washington	Pullman, Wash.	Apr. 27–29
Montana State University	Missoula, Mont.	May 1–5
Utah State University	Logan, Utah	May 8–11
Colorado State University	Fort Collins, Colo.	May 14–15
University of Colorado	Boulder, Colo.	May 18–20
Los Angeles State College	Los Angeles, Calif.	To be arranged
Pomona College	Claremont, Calif.	To be arranged
Whittier College	Whittier, Calif.	To be arranged

Robert E. Gaskell

University of Alberta	Edmonton, Canada	Jan. 5–6
University of Minnesota	Duluth 11, Minn.	Jan. 7–8
Wisconsin State College	Superior, Wis.	Jan. 9
University of North Dakota	Grand Forks, N. D.	Jan. 12–13
North Dakota Agricultural College	Fargo, N. D.	Jan. 14–15
Concordia College	Moorhead, Minn.	Jan. 16
Grinnell College	Grinnell, Iowa	Jan. 19–20
Baldwin-Wallace College	Berea, Ohio	Jan. 28–30
Antioch College	Yellow Springs, Ohio	Feb. 2–4
University of Dayton	Dayton, Ohio	Feb. 4–6
University of Buffalo	Buffalo 14, N. Y.	Feb. 9–10
University of Vermont	Burlington, Vt.	Feb. 11–12
Norwich University	Northfield, Vt.	Feb. 12–13
Hamilton College	Clinton, N. Y.	Feb. 16–17
Alfred University	Alfred, N. Y.	Feb. 18–19
University of Detroit	Detroit 21, Mich.	Feb. 19–20
University of Kansas	Lawrence, Kan.	Feb. 23–24
Ottawa University	Ottawa, Kan.	Feb. 25
St. Benedict's College and Mount St. Scholastica College	Atchison, Kan.	Feb. 26–27
Iowa State Teachers College	Cedar Falls, Iowa	Mar. 2–3
Carroll College	Waukesha, Wis.	Mar. 5–6
Purdue University	Indianapolis, Ind.	Mar. 9
Butler University	Indianapolis 7, Ind.	Mar. 10–11
Valparaiso University	Valparaiso, Ind.	Mar. 12–13
Albion College	Albion, Mich.	Mar. 16
North Central College	Naperville, Ill.	Mar. 18
University of Wisconsin	Milwaukee, Wis.	Mar. 19–20
University of Omaha	Omaha, Neb.	Mar. 23–24
Indiana Technical College	Fort Wayne, Ind.	Mar. 26–27
Drake University	Des Moines, Iowa	Mar. 30–31

R. E. Johnson

Randolph-Macon Women's Coll. and Lynchburg Coll.	Lynchburg, Va.	Feb. 3–4
Sweet Briar College	Sweet Briar, Va.	Feb. 3–4
University of Richmond	Richmond, Va.	Feb. 5–6
College of William and Mary	Norfolk, Va.	Feb. 9–10
East Carolina College	Greenville, N. C.	Feb. 12–13
University of North Carolina	Chapel Hill, N. C.	Feb. 16–17
Carolina State College and Duke University	Durham, N. C.	Feb. 16–17
North Georgia College	Dahlonega, Ga.	Feb. 19–20
Alabama Polytechnic Institute	Auburn, Ala.	Mar. 2–3
Mississippi State University	State College, Miss.	Mar. 5–6
University of Mississippi	University, Miss.	Mar. 9–10
Southwestern Louisiana Institute	Lafayette, La.	Mar. 12–13
Centenary College	Shreveport, La.	Mar. 16–17
University of Arkansas	Fayetteville, Ark.	Mar. 19–21
Washington University	St. Louis, Mo.	Mar. 30–31
University of Wichita	Wichita, Kan.	Apr. 2–3
Southwestern at Memphis	Memphis, Tenn.	Apr. 8–10
University of Kentucky	Lexington, Ky.	Apr. 13–15
Wilson College	Chambersburg, Pa.	Apr. 17
State Teachers College	Shippensburg, Pa.	Apr. 20–24
Dickinson College	Carlisle, Pa.	Apr. 20–24
Bucknell University	Lewisburg, Pa.	May 4–6
Hartwick College and State University Teachers College	Oneonta, N. Y.	May 7–8
Upper New York Section of M.A.A.	Oneonta, N. Y. (Hartwick Coll.)	May 9
Montclair State College	Upper Montclair, N. J.	May 12–13
St. John's University	Jamaica, N. Y.	May 14–15

John L. Kelley

University of Miami	Coral Gables, Fla.	Apr. 1–3
University of Florida	Gainesville, Fla.	Apr. 6–7
Florida A. and M. University and Florida State University	Tallahassee, Fla.	Apr. 13–17
Georgia Institute of Technology	Atlanta, Ga.	Apr. 20–24

Stanislaw Ulam

California Institute of Technology	Pasadena, Calif.	Jan. 7
Occidental College	Los Angeles, Calif.	Jan. 8–9
Univ. of California at Santa Barbara	Goleta, Calif.	Jan. 12–13
Oregon State College	Corvallis, Ore.	Jan. 15–16
Reed College	Portland, Ore.	Jan. 19–20
University of Utah	Salt Lake City, Utah	Jan. 22
Brigham Young University	Provo, Utah	Jan. 23
Univ. of New Mexico	Albuquerque, N. M.	To be arranged

S. S. Wilks

East Texas State College	Commerce, Tex.	Oct. 30–31
Oklahoma State University	Stillwater, Okla.	Nov. 3–7
Oklahoma Central State College	Edmond, Okla.	Nov. 3–7
North Texas State College	Denton, Tex.	Apr. 8–18

Southern Methodist University	Dallas, Tex.	Apr. 8–18
Texas Christian University	Fort Worth, Tex.	Apr. 8–18
Rice Institute	Houston, Tex.	Apr. 8–18

THE HIGH SCHOOL MATHEMATICS CONTEST

The Mathematical Association of America, in cooperation with the Society of Actuaries, this past spring assumed control of the high school contest examination which had been administered for the last few years by the Committee on Contests and Awards of the Metropolitan New York Section of the Mathematical Association of America.

On March 27, 1958, an eighty-minute test, consisting of fifty questions, was given to over 80,000 students in 2889 high schools. These figures compare with 43,500 students in 1469 high schools in 1957 and show this year's truly explosive growth in both number of participating schools and contestants. Nearly all of the states of the United States were represented, many in large numbers, as were also a number of provinces of Canada, and Alaska, Hawaii, American Samoa, and the Canal Zone.

The contest for 1959 will be held on Thursday morning, March 5, 1959, during the first two class periods. The examination will be of the same type as was used for the 1958 contest. High schools who have not as yet received their registration form for this contest should contact the contest chairman of their section of the M.A.A. or Professor W. H. Fagerstrom, Executive Director of the Contest, Pan American College, Edinburg, Texas.

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The third annual meeting of the New Jersey Section of the Mathematical Association of America was held at Rutgers, The State University, New Brunswick, New Jersey, on November 1, 1958. Professor B. E. Meserve, Chairman of the Section, presided at the morning session, and Dr. H. O. Pollak, Senior-Member-at-Large of the Executive Committee, presided at the afternoon session. There were 135 persons present, including 100 members of the Association.

The following new officers were elected: Chairman, Professor S. S. Wilks, Princeton University; Member-at-Large of the Executive Committee, Professor E. P. Starke, Rutgers, The State University. It was noted that Dr. H. O. Pollak will be Chairman of the Program Committee during the coming year, and that Dr. J. D. Daugherty, Eastside High School, Paterson, continues as a Member-at-Large of the Executive Committee.

Dr. George Cherlin, Secretary of the Contest Committee, reported that the contest had been very successful, with 127 schools enrolled and about 3500 students participating.

The following invited addresses of one hour each were presented at the morning session:

1. *The mathematical theory of strictly competitive games*, by Dr. Harlan Mills, Market Research Corporation of America and Princeton University.

Strictly competitive games, exemplified by matching pennies, and poker, provide two mathematical problems: 1) restating their rules in "normalized" form—given a matrix $G = [g_{ij}]$, players I, II simultaneously pick strategies s, t and II pays g_{st} to I; 2) finding probability mixtures of strategies which guarantee the players as favorable expectations as possible, solving problems $\max_x \min_y xGy$ and $\min_y \max_x xGy$, x and y probability vectors—these can be restated as dual linear programs. These best guarantees are identical, and this supplies persuasive rationale for adopting the probability mixtures—the simplex method places them within computational reach.

2. *Linear programming*, by Dr. A. J. Hoffman, General Electric Company, New York, New York.

The main part of the address was a discussion of three applications of linear programming

The first application was the determination of the most profitable product mix of a boot factory. The second discussed the award of contracts at least cost to the awarder. The third application was the exploitation of the duality theorem of linear programming to yield a combinatorial theorem. The address concluded with an explanation of two geometric interpretations of the simplex method.

The program at the afternoon session consisted of a panel discussion on

3. *Teacher education for modern mathematics.*

Dean A. E. Meder, Jr., Rutgers, The State University, who had arranged this part of the program, presented the members of the panel, and introduced the subject. Professor A. W. Tucker, Princeton University, presided.

Each member of the panel discussed one facet of the subject: Professor F. G. Fender, Rutgers, The State University, spoke on computers; Professor E. R. Lorch, Columbia University, spoke on algebra; Professor B. E. Meserve, New Jersey State College, Upper Montclair, spoke on geometry; Professor R. M. Walter, Douglass College, Rutgers, The State University, spoke on probability and statistical inference; and Professor Tucker concluded the formal presentations with a discussion of analysis. There was then discussion from the floor.

Since the formal presentations were recorded and will appear under Mathematical Education Notes in this MONTHLY, the abstracts are not included here.

I. L. BATTIN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Hotel Conrad Hilton, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Valparaiso University, May 2, 1959.

IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.

KANSAS, Marymount College, Salina, April 11, 1959.

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, May 2, 1959.

METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.

MICHIGAN, Michigan State University, East Lansing, March 28, 1959.

MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Miami University, Oxford, May 9, 1959.

OKLAHOMA, Tulsa University, Spring, 1959.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA

ROCKY MOUNTAIN, Utah State University, Logan, May 8–9, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State University, Tempe, April 10–11, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May 2, 1959.

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